The formal KZ equation on the moduli space $\mathcal{M}_{0,5}$ and the harmonic product of multiple zeta values

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Abstract

In this article, we will show that a normalized fundamental solution of the formal Knizhnik-Zamolodchikov equation on the moduli space $\mathcal{M}_{0,5}$ are decomposed in two ways, and that each decomposition comes from iterated integral along specified integral contours. Hence the normalized fundamental solution has iterated integral expressions of two types. Comparing them, one has the generalized harmonic product relations for hyperlogarithms of type $\mathcal{M}_{0,5}$. These relations constitute a bigger class that properly involves the harmonic product of multiple polylogarithms. From this, we obtain the harmonic product of multiple zeta values.

1 Introduction

The formal Knizhnik-Zamolodchikov equation on the moduli space $\mathcal{M}_{0,n}$ of configuration of n points in $\mathbf{P}^1$ is an integrable pfaffian system with coefficients in the infinitesimal pure braid Lie algebra $\mathfrak{X}$.

In the case of $\mathcal{M}_{0,4}$, the formal KZ equation is a Fuchsian differential equation in the variable z, with regular singular points at $z = 0, 1, \infty$, and the Lie algebra $\mathfrak{X}$ is a free Lie algebra generated by two elements, say X_0, X_1. The formal KZ equation on $\mathcal{M}_{0,4}$ is referred to as the formal KZ equation of one variable. The fundamental solution normalized at $z = 0$ is a grouplike element of $\mathbf{C}\langle\langle X_0, X_1\rangle\rangle$ which is a completion of the universal enveloping algebra $\mathcal{U}(\mathfrak{X})$ of $\mathfrak{X}$, and can be regarded as a generating function of multiple polylogarithms of one variable

$$\mathrm{Li}_{k_1,\ldots,k_r}(z) = \sum_{n_1>n_2>\cdots>n_r>0} \frac{z^{n_1}}{n_1^{k_1}\cdots n_r^{k_r}}.$$

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The connection matrix of the fundamental solutions normalized at $z=0$ and $z=1$ of the formal KZ equation of one variable is known as the Drinfel'd associator, and is denoted by $\Phi_{\mathrm{KZ}}(X_0, X_1)$. It is a grouplike element of $\mathbf{C}\langle\langle X_0, X_1\rangle\rangle$, and is thought of to be a generating function for multiple zeta values

$$\zeta(k_1, \ldots, k_r) = \sum_{n_1>n_2>\cdots>n_r>0} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}}$$

[D],[LM]. The Drinfel'd associator enjoys the duality relation, the hexagon relation and the pentagon relation. An associator is, by definition, a grouplike elements of $\mathbf{C}\langle\langle X_0, X_1\rangle\rangle$ satisfying these three relations. The Grothendieck-Teichmüller group of associators is closely related to the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ [D],[I].

That the Drinfel'd associator $\Phi_{\mathrm{KZ}}(X_0, X_1)$ satisfies the duality relation and the hexagon relation is shown by considering the connection problem for the formal KZ equation of one variable [K], [OkU]. Furthermore, that the totality of multiple zeta values is closed under the shuffle product is equivalent to that $\Phi_{\mathrm{KZ}}(X_0, X_1)$ is a grouplike element of $\mathbf{C}\langle\langle X_0, X_1\rangle\rangle$.

There is the second way to define the product of multiple zeta values; it is the harmonic product or the series shuffle product. It is a generalization of the typical example

$$\begin{aligned}\zeta(k)\zeta(l) &= \sum_{m>0}\frac{1}{m^k}\sum_{n>0}\frac{1}{n^l} = \left(\sum_{m>n>0} + \sum_{m=n>0} + \sum_{n>m>0}\right)\frac{1}{m^k n^l}\\ &= \zeta(k,l) + \zeta(k+l) + \zeta(l,k)\end{aligned}$$

(as for the precise definition, see Section 8.1).

Equating the harmonic product and the shuffle product of multiple zeta values, we obtain nontrivial $\mathbf{Q}$-linear relations for multiple zeta values known as the double shuffle relation [IKZ]. It is conjectured that the (regularized) double shuffle relations contain all the $\mathbf{Q}$-linear relations for multiple zeta values.

Deligne-Terasoma [DT], Besser-Furusho [BF] and Furusho [F] derived the double shuffle relations from the pentagon relation for the associators by means of methods in arithmetic geometry. On the other hand, that the Drinfel'd associator satisfies the pentagon relation is shown by considering the connection problem for the formal KZ equation on $\mathcal{M}_{0,5}$ (the formal KZ equation of two variables) [W], [OU1]. So one can expect that there is a direct way to obtain the harmonic product of multiple zeta values.

In this article, we will derive the harmonic product of multiple zeta values from the decomposition theorem of the normalized fundamental solution of the formal KZ equation of two variables.

The formal KZ equation of two variables is an integrable pfaffian system on $\mathbf{P}^1 \times \mathbf{P}^1$ with logarithmic poles along the divisors $D = \{z_1 = 0, 1, \infty\} \cup \{z_2 = 0, 1, \infty\} \cup \{z_1 z_2 = 1\}$. The fundamental solution normalized at the origin is a grouplike element of $\widetilde{\mathcal{U}}(\mathfrak{X})$, a completion of the universal enveloping algebra $\mathcal{U}(\mathfrak{X})$, and can be viewed as a generating function of hyperlogarithms of type $\mathcal{M}_{0,5}$

$$L({}^{k_1}\alpha_1 \cdots {}^{k_r}\alpha_r; z_1) = \sum_{n_1>n_2>\cdots>n_r>0} \frac{\alpha_1^{n_1-n_2}\alpha_2^{n_2-n_3}\cdots\alpha_r^{n_r}}{n_1^{k_1}\cdots n_r^{k_r}} z_1^{n_1},$$

where $\alpha_i = 1$ or $\alpha_i = z_2$. The decomposition theorem (Section 6.2) says that the normalized fundamental solution $\mathcal{L}(z_1, z_2)$ decomposes, in two ways, to product of the normalized fundamental solutions of the formal (generalized) KZ equations of one variable:

$$\mathcal{L}(z_1, z_2) = \mathcal{L}^{(1)}_{1\otimes 2}\mathcal{L}^{(2)}_{1\otimes 2} = \mathcal{L}^{(2)}_{2\otimes 1}\mathcal{L}^{(1)}_{2\otimes 1}.$$

This theorem plays a basic role in the connection problem of the formal KZ equation of two variables. Furthermore, one can show that each decomposition comes from iterated integrals along specified integral contours $C_{1\otimes 2}$ and $C_{2\otimes 1}$ (Section 7.2). Hence the normalized fundamental solution has iterated integral expressions of two types. Equating them, one has the generalized harmonic product relations for hyperlogarithms of type $\mathcal{M}_{0,5}$. Furthermore it is shown that they involve properly the harmonic product of multiple polylogarithms, and taking the limit of them, the harmonic product of multiple zeta values is obtained.

The connection problem and the transformation theory of the formal KZ equation on $\mathcal{M}_{0,n}$ for general n will appear in our forthcoming paper [OU2].

The paper is organized as follows: In Section 2, we give an account for the formal KZ equation on the moduli space $\mathcal{M}_{0,n}$ in the simplicial coordinate system and in the cubic coordinate system. Especially, the infinitesimal pure braid relations and the Arnold relations for differential forms in each coordinate system are considered in details. The formal KZ equation represented in the cubic coordinate system is referred to as the formal KZ equation of $n-3$ variables. In Section 3, we consider the formal KZ equation of one variable (the formal 1KZ equation, for short). Representing the normalized fundamental solution in terms of iterated integral, we see that it is viewed as a generating function of multiple polylogarithms of one variable. We also consider the formal generalized 1KZ equation, and show that the normalized fundamental solution is thought of to be a generating function of hyperlogarithms of general type. In Section 4, the formal KZ equation of two variables (the formal 2KZ equation, for short) is introduced and the tensor product decomposition theorem for the universal enveloping algebra $\mathcal{U}(\mathfrak{X})$ of the infinitesimal pure braid Lie algebra $\mathfrak{X}$ (Proposition 2) is established.

To solve the formal 2KZ equation by means of iterated integrals, in Section 5 we consider an algebra $\mathcal{B}$ of differential forms which is called the reduced bar algebra. Such an algebra of differential forms is a standard one in the theory of the loop space cohomology [C1], [C2], and the reduced bar algebra is nothing but the 0-th cohomology of the reduced bar construction of the Orlik-Solomon algebra associated with the hyperplanes arrangement of type A_4 [OT]. One can regard $\mathcal{B}$ as a dual Hopf algebra of $\mathcal{U}(\mathfrak{X})$ (Proposition 4), and through this duality, we can prove the tensor product decomposition for $\mathcal{B}$ (Lemma 7 and Proposition 9). For an element of $\mathcal{B}$, we define the iterated integral [C3] and the value of the integral depends on only the homotopy type of the integral contour. The tensor product decomposition theorem for $\mathcal{B}$ says that, choosing two specific integral contours $C_{1\otimes 2}$ and $C_{2\otimes 1}$, the iterated integral of two variables reduces to the product of iterated integrals of one variable in two ways. We also note that the iterated integral is injective by the results of [B]. Therefore relations of iterated integrals are equivalent to relations of elements in $\mathcal{B}$.

In Section 6, we first prove the existence and the uniqueness of a normalized fundamental solution (Proposition 14), and next the decomposition theorem for the fundamental solution (Proposition 16). In Section 7, we show that the decomposition in two ways comes from the iterated integral expression along $C_{1\otimes 2}$ and $C_{2\otimes 1}$. As a result, the comparison of the iterated integrals appearing in the coefficients of the two decomposition of the fundamental solution leads to the generalized harmonic product relations for hyperlogarithms of type $\mathcal{M}_{0,5}$ (Theorem 22).

In Section 8, we show that the generalized harmonic product relations properly involve the harmonic product of multiple polylogarithms (Theorem 27).

Thus the harmonic product of the multiple zeta values is interpreted from the viewpoint of the decomposition theorem of the normalized fundamental solution of the formal 2KZ equation.

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2 The formal KZ equation on the moduli space $\mathcal{M}_{0,n}$

First we define the formal KZ equation on the configuration space $(\mathbf{P}^1)^n_*$, and show that it can be viewed as a differential equation on the moduli space $\mathcal{M}_{0,n}$. On the moduli space, introducing the simplicial coordinates and the cubic coordinates, we consider the formal KZ equation in these coordinate systems.

2.1 The formal KZ equation on the configuration space $(\mathbf{P}^1)^n_*$

On the configuration space of n points in $\mathbf{P}^1$

$$(\mathbf{P}^1)^n_* = \{(x_1,\ldots,x_n) \in \underbrace{\mathbf{P}^1 \times \cdots \times \mathbf{P}^1}_{n} \mid x_i \neq x_j \;\; (i \neq j)\}, \tag{1}$$

we consider a pfaffian system

$$dG = \varOmega G, \quad \text{where} \quad \varOmega = \sum_{i<j} d\log(x_i - x_j)X_{ij}. \tag{2}$$

Here X_{ij}'s are formal elements satisfying the infinitesimal pure braid relations (IPBR, for short)

$$\sum_{j=1}^{n} X_{ij} = 0 \quad (1 \leq i \leq n), \quad X_{ij} = X_{ji},\; X_{ii} = 0 \quad (1 \leq i,j \leq n), \tag{3}$$

$$[X_{ij}, X_{kl}] = 0 \quad (\{i,j\} \cap \{k,l\} = \emptyset). \tag{4}$$

The Lie algebra generated by X_{ij}'s with the fundamental relations IPBR will be denoted by $\mathfrak{X}$ and referred to as the infinitesimal pure braid Lie algebra (IPB Lie algebra, for short).

Set $\xi_{ij} = d\log(x_i - x_j) \quad (i \neq j)$ and $\xi_{ii} = 0$. Note that $\xi_{ij} = \xi_{ji}$.

The system is integrable and is invariant under the diagonal action of $\mathrm{PGL}(2, \mathbf{C})$ on $(\mathbf{P}^1)^n_*$. In fact, using only the trivial relations $\xi_{ij} \wedge \xi_{kl} + \xi_{kl} \wedge \xi_{ij} = 0$, we have

$$\begin{aligned}\Omega \wedge \Omega = & \sum_{\substack{i<j,k<l \\ \{i,j\}\cap\{k,l\}=\emptyset}} \xi_{ij} \wedge \xi_{kl}[X_{ij}, X_{kl}] + \sum_{i<j<k} \xi_{ij} \wedge \xi_{ik}[X_{ij}, X_{ik}] + \\ & + \sum_{i<j<k} \xi_{ik} \wedge \xi_{jk}[X_{ik}, X_{jk}] + \sum_{i<j<k} \xi_{ij} \wedge \xi_{jk}[X_{ij}, X_{jk}].\end{aligned}$$

Lemma 1 ([A]). The nontrivial relations between the one-forms $\{\xi_{ij}\}$ are only

$$\xi_{ij} \wedge \xi_{ik} + \xi_{ik} \wedge \xi_{jk} + \xi_{jk} \wedge \xi_{ij} = 0. \tag{5}$$

We call (5) the Arnold relations (AR, for short). From this lemma, we obtain

$$\begin{aligned}\Omega \wedge \Omega = & \sum_{\substack{i<j,k<l \\ \{i,j\}\cap\{k,l\}=\emptyset}} \xi_{ij} \wedge \xi_{kl}[X_{ij}, X_{kl}] + \\ & + \sum_{i<j<k} \xi_{ij} \wedge \xi_{ik}[X_{ij} + X_{jk}, X_{ik}] + \sum_{i<j<k} \xi_{ij} \wedge \xi_{jk}[X_{ij} + X_{ik}, X_{jk}].\end{aligned}$$

Hence the integrable condition $\Omega \wedge \Omega = 0$ is equivalent to

$$[X_{ij}, X_{kl}] = 0 \quad (\{i,j\} \cap \{k,l\} = \emptyset), \quad [X_{ij} + X_{jk}, X_{ik}] = 0 \quad (\#\{i,j,k\} = 3), \tag{6}$$

which is deduced from (3) and (4).

Furthermore, by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2, \mathbf{C})$, Ω is transformed to

$$\Omega - \sum_{i=1}^{n} d\log(cx_i + d)\Big(\sum_{j=1}^{n} X_{ij}\Big) + d\log(ad - bc)\Big(\sum_{1\leq i<j\leq n} X_{ij}\Big).$$

From (3), we have $\sum_{j=1}^{n} X_{ij} = 0$ and $\sum_{1\leq i<j\leq n} X_{ij} = 0$ so that the equation is $\mathrm{PGL}(2, \mathbf{C})$-invariant. Thus it yields an integrable system on the moduli space $\mathcal{M}_{0,n} = \mathrm{PGL}(2, \mathbf{C})\backslash(\mathbf{P}^1)^n_*$.

2.2 The formal KZ equation in the simplicial coordinate system

Introduce the coordinate system $(y_1, \ldots, y_{n-3})$ on the moduli space $\mathcal{M}_{0,n}$ via

$$y_i = \frac{x_i - x_{n-2}}{x_i - x_n} \cdot \frac{x_{n-1} - x_n}{x_{n-1} - x_{n-2}} \quad (1 \leq i \leq n-3), \quad (y_{n-2} = 0,\ y_{n-1} = 1,\ y_n = \infty). \tag{7}$$

We call (7) the simplicial coordinate system due to Brown [B]. In this coordinate system, the formal KZ equation takes the following form:

$$dG = \Big\{\sum_{k=1}^{n-3} d\log y_k Y_{k,n-2} + \sum_{k=1}^{n-3} d\log(y_k - 1) Y_{k,n-1} + \sum_{1\leq i<j\leq n-3} d\log(y_i - y_j) Y_{ij}\Big\} G. \tag{8}$$

Here we set $Y_{ij} = X_{ij}$ $(1 \leq i, j \leq n-3), Y_{k,n-2} = X_{k,n-2}, Y_{k,n-1} = X_{k,n-1}$. The equation can be rewritten in an equivalent form

$$\frac{\partial G}{\partial y_k} = \Big(\frac{Y_{k,n-2}}{y_k} + \frac{Y_{k,n-1}}{y_k - 1} + \sum_{1 \leq j (\neq k) \leq n-3} \frac{Y_{kj}}{y_k - y_j}\Big) G \qquad (k = 1, \ldots, n-3). \quad (9)$$

The IPBR (3), (4) for the generators $\{Y_{ij}\}_{1 \leq i,j \leq n-1}$ reads

$$\begin{cases} Y_{ij} = Y_{ji}, \quad Y_{ii} = 0, \quad Y_{n-2,n-1} = Y_{n-1,n-2} = 0, \\ [Y_{ij}, Y_{kl}] = 0 \quad (\{i,j\} \cap \{k,l\} = \emptyset), \\ [Y_{ij} + Y_{jk}, Y_{ik}] = 0 \quad (\#\{i,j,k\} = 3), \end{cases} \quad (10)$$

and, setting $\eta_{ij} = d\log(y_i - y_j)$ $(1 \leq i \neq j \leq n-1)$, $\eta_{ii} = \eta_{n-2,n-1} = \eta_{n-1,n-2} = 0$ $(1 \leq i \leq n-1)$, the AR (5) reads

$$\begin{cases} \eta_{ij} \wedge \eta_{ik} + \eta_{ik} \wedge \eta_{jk} + \eta_{jk} \wedge \eta_{ij} = 0 & (1 \leq i \neq j \leq n-3), \\ \eta_{ij} \wedge \eta_{i,n-2} + \eta_{i,n-2} \wedge \eta_{j,n-2} + \eta_{j,n-2} \wedge \eta_{ij} = 0 & (1 \leq i \neq j \leq n-3), \\ \eta_{ij} \wedge \eta_{i,n-1} + \eta_{i,n-1} \wedge \eta_{j,n-1} + \eta_{j,n-1} \wedge \eta_{ij} = 0 & (1 \leq i \neq j \leq n-3), \\ \eta_{i,n-2} \wedge \eta_{i,n-1} = 0 & (1 \leq i \leq n-3). \end{cases} \quad (11)$$

2.3 The formal KZ equation in the cubic coordinate system

Introduce the coordinate system $(z_1, \ldots, z_{n-3})$ on the moduli space $\mathcal{M}_{0,n}$ through

$$z_1 = y_1, z_2 = \frac{y_2}{y_1}, \cdots, z_{n-3} = \frac{y_{n-3}}{y_{n-4}} \iff y_i = z_1 \cdots z_i \quad (1 \leq i \leq n-3). \quad (12)$$

In this coordinate system, the formal KZ equation is represented as follows:

$$dG = \Big\{\sum_{k=1}^{n-3} \zeta_k Z_k + \sum_{k=1}^{n-3} \zeta_{kk} Z_{kk} + \sum_{1 \leq i < j \leq n-3} \zeta_{ij} Z_{ij}\Big\} G. \quad (13)$$

Here we set

$$\begin{cases} Z_k = \sum_{k \leq i < j \leq n-2} X_{ij} & (1 \leq k \leq n-3), \\ Z_{11} = -X_{1,n-1}, \quad Z_{kk} = -X_{k-1,k} & (2 \leq k \leq n-3), \\ Z_{1j} = -X_{j,n-1} \quad (2 \leq j \leq n-3), \quad Z_{ij} = -X_{i-1,j} & (2 \leq i < j \leq n-3), \end{cases} \quad (14)$$

and

$$\zeta_i = \frac{dz_i}{z_i}, \quad \zeta_{ij} = \frac{d(z_i \cdots z_j)}{1 - z_i \cdots z_j} \quad (i \leq j). \quad (15)$$

We call (13) the formal KZ equation of $n-3$ variables (in the cubic coordinate system). This is equivalent to

$$\frac{\partial G}{\partial z_k} = \Big(\frac{Z_k}{z_k} + \frac{Z_{kk}}{1 - z_k} + \sum_{\substack{1 \leq i < j \leq n-3 \\ i \leq k \leq j}} \frac{z_{ij}^{(k)} Z_{ij}}{1 - z_{ij}}\Big) G \qquad (1 \leq k \leq n-3), \quad (16)$$

where $z_{ij} = z_i \cdots z_j$ and $z_{ij}^{(k)} = z_{ij}/z_k \ \ (i \le k \le j)$.

The IPBR for the generators $\{Z_i, Z_{ii}\ (1 \le i \le n{-}3),\ Z_{ij}\ (1 \le i < j \le n{-}3)\}$ are

$$
\begin{cases}
[Z_i, Z_j] = 0 & (\forall i, j), \\
[Z_i, Z_{jj}] = 0 & (i \ne j), \\
[Z_i, Z_{jk}] = 0 & (i < j < k \text{ or } j < k < i), \\
[Z_{ij}, Z_{jk}] = 0 & (i < j < k), \\
[Z_{ij}, Z_{lk}] = 0 & (i < l \le k < j \text{ or } i < l < j < k \\
 & \qquad \text{or } i < j, l \le k, 1 < l - j), \\
[Z_{ii}, Z_{i+1,j}] = -[Z_{ii}, Z_{i,j}] = [Z_{i+1,j}, Z_{i,j}] = -[Z_i - Z_{i+1}, Z_{i,j}] & \\
 & (1 \le i < i+1 < j \le n-3), \\
[Z_{ij}, Z_{j+1,j+1}] = -[Z_{i,j+1}, Z_{j+1,j+1}] = [Z_{i,j+1}, Z_{ij}] = -[Z_j - Z_{j+1}, Z_{i,j+1}] & \\
 & (1 \le i < j <\le n-4).
\end{cases}
\tag{17}
$$

The AR for the forms $\{\zeta_i, \zeta_{ii}\ (1 \le i \le n-3),\ \zeta_{ij}\ (1 \le i < j \le n-3)\}$ reads

$$
\begin{cases}
\zeta_{i+1,j} \wedge \zeta_{i+1,k} + \zeta_{i+1,k} \wedge (\zeta_{j+1,k} - \zeta_{i+1} - \cdots - \zeta_j) & \\
\qquad + (\zeta_{j+1,k} - \zeta_{i+1} - \cdots - \zeta_j) \wedge \zeta_{i+1,j} = 0 & \\
 & (1 \le i < j < k \le n-3), \\
(\zeta_{i+1} + \cdots + \zeta_j) \wedge \zeta_{i+1,j} = 0 & (1 \le i < j \le n-3), \\
\zeta_{i+1,j} \wedge \zeta_{1i} + \zeta_{1i} \wedge \zeta_{1j} + \zeta_{1j} \wedge (\zeta_{i+1,j} + \zeta_{i+1} + \cdots + \zeta_j) = 0 & \\
 & (1 \le i < j \le n-3), \\
(\zeta_1 + \cdots + \zeta_i) \wedge \zeta_{1i} = 0 & (1 \le i \le n-3).
\end{cases}
\tag{18}
$$

3 The one variable theory

3.1 The formal KZ equation of one variable

The formal KZ equation of one variable will be referred to as the formal 1KZ equation for short. It is represented as follows:

$$dG = \Omega G, \quad \Omega = \zeta_1 Z_1 + \zeta_{11} Z_{11}. \tag{19}$$

Here we set $z = z_1$ and

$$Z_1 = X_{12}, \quad Z_{11} = -X_{13}, \quad \zeta_1 = \frac{dz}{z}, \quad \zeta_{11} = \frac{dz}{1-z}.$$

The formal 1KZ equation is a Fuchsian differential equation on $\mathbf{P}^1$ with regular singular points $0, 1, \infty$.

The IPBR for $\{Z_1, Z_{11}\}$ is trivial so that the IPB Lie algebra $\mathfrak{X}$ is a free Lie algebra. That is, $\mathfrak{X} = \mathbf{C}\{Z_1, Z_{11}\}$. The universal enveloping algebra $\mathcal{U}(\mathfrak{X})$ is the ring of non-commutative polynomials of the variables Z_1, Z_{11}. Namely $\mathcal{U}(\mathfrak{X}) = \mathbf{C}\langle Z_1, Z_{11}\rangle$. The unit element of $\mathcal{U}(\mathfrak{X})$ will be denoted by $\mathbf{I}$.

The Lie algebra $\mathfrak{X}$ has grading with respect to the degree of Lie polynomials of Z_1, Z_{11}, and is a graded Lie algebra:

$$\mathfrak{X} = \bigoplus_{s=1}^{\infty} \mathfrak{X}_s, \qquad [\mathfrak{X}_s, \mathfrak{X}_{s'}] \subset \mathfrak{X}_{s+s'}. \tag{20}$$

The universal enveloping algebra $\mathcal{U}(\mathfrak{X})$ has also grading with respect to the length of words of Z_1, Z_{11}:

$$\mathcal{U}(\mathfrak{X}) = \bigoplus_{s=0}^{\infty} \mathcal{U}_s(\mathfrak{X}), \qquad \mathcal{U}_s(\mathfrak{X}) \cdot \mathcal{U}_{s'}(\mathfrak{X}) \subset \mathcal{U}_{s+s'}(\mathfrak{X}). \tag{21}$$

The completion of $\mathcal{U}(\mathfrak{X})$ with respect to this grading will be denoted as $\widetilde{\mathcal{U}}(\mathfrak{X})$. This is nothing but the ring of non-commutative formal power series of the variables Z_1, Z_{11}. That is, $\widetilde{\mathcal{U}}(\mathfrak{X}) = \mathbf{C}\langle\langle Z_1, Z_{11}\rangle\rangle$. A solution to the formal 1KZ equation can be regarded as a function which takes the values in $\widetilde{\mathcal{U}}(\mathfrak{X})$.

The dual Hopf algebra of $\mathcal{U}(\mathfrak{X})$ is a free shuffle algebra [R] $S(A)$ generated by an alphabet $A = \{\xi_1, \xi_{11}\}$. The free shuffle algebra $S(A)$ is a non-commutative polynomial ring $\mathbf{C}\langle \xi_1, \xi_{11}\rangle$ of the variables ξ_1, ξ_{11}. In this algebra we put the shuffle product in use, and the multiplication as the non-commutative polynomial ring (namely, the concatenation) is denoted by $\circ$. We will omit it if there is no worry of confusing. $\mathbf{1}$ stands for the unit element in $S(A)$ which corresponds to the null word. The shuffle product is defined recursively as follows:

$$\mathbf{1} \sqcup\!\sqcup\, w = w \sqcup\!\sqcup\, \mathbf{1} = \mathbf{1}, \tag{22}$$

$$(a_1 \circ w_1) \sqcup\!\sqcup\, (a_2 \circ w_2) = a_1 \circ (w_1 \sqcup\!\sqcup\, (a_2 \circ w_2)) + a_2 \circ ((a_1 \circ w_1) \sqcup\!\sqcup\, w_2). \tag{23}$$

Here $a_1, a_2 \in A$, and w, w_1, w_2 are words in $S(A)$. Then $S(A) = (\mathbf{C}\langle \xi_0, \xi_1\rangle, \sqcup\!\sqcup)$ is an associative commutative algebra [R].

The AR for the forms $\{\zeta_1, \zeta_{11}\}$ is just $\zeta_1 \wedge \zeta_{11} = 0$, which is the last relation in (18) of $n = 4$ and trivial in this case. Let us identify

$$\xi_1 = \zeta_1, \qquad \xi_{11} = \zeta_{11}. \tag{24}$$

Under this identification, we denote the free shuffle algebra by $S(\zeta_1, \zeta_{11})$. For a word $\omega = \omega_1 \circ \cdots \circ \omega_r \in S(\zeta_1, \zeta_{11})$ where $\omega_i = \zeta_1$ or ζ_{11}, define an iterated integral $\int \omega$ by

$$\int \mathbf{1} = 1, \tag{25}$$

$$\int_{z_0}^{z} \omega = \int_{z_0}^{z} \omega_1(z') \int_{z_0}^{z'} \omega_2 \circ \cdots \circ \omega_r. \tag{26}$$

This integral defines a many valued analytic function on $\mathbf{P}^1 - \{0, 1, \infty\}$ and, at the same time, defines a homomorphism from $S(\zeta_1, \zeta_{11})$ to $\mathbf{C}$, namely we have

$$\int (\omega_1 \sqcup\!\sqcup\, \omega_2) = \left(\int \omega_1\right)\left(\int \omega_2\right) \tag{27}$$

for any words in $S(\zeta_1, \zeta_{11})$. An element in $S(\zeta_1, \zeta_{11})$ will be called an iterated form.

Next we consider the fundamental solution of the formal 1KZ equation normalized at the origin $z = 0$. Denote it by $\mathcal{L}(z)$. It is a solution satisfying the following condition:

$$\mathcal{L}(z) = \hat{\mathcal{L}}(z) z^{Z_1} \tag{28}$$

where $\hat{\mathcal{L}}(z)$ is a function which takes the values in $\widetilde{\mathcal{U}}(\mathfrak{X})$, and holomorphic at $z = 0$, and satisfies $\hat{\mathcal{L}}(0) = \mathbf{I}$. Under the gradation of $\mathcal{U}(\mathfrak{X})$ (21),we can represent $\hat{\mathcal{L}}(z)$ as

$$\hat{\mathcal{L}}(z) = \sum_{s=0}^{\infty} \hat{\mathcal{L}}_s(z), \quad \hat{\mathcal{L}}_s(z) \in \mathcal{U}_s(\mathfrak{X}), \quad \hat{\mathcal{L}}_s(0) = 0 \ \ (s > 0), \quad \hat{\mathcal{L}}_0(z) = \mathbf{I}. \tag{29}$$

It is easy to see that $\hat{\mathcal{L}}_s(z)$ satisfies the following recursive equation:

$$\frac{d\hat{\mathcal{L}}_{s+1}}{dz} = \frac{1}{z}[Z_1, \hat{\mathcal{L}}_s] + \frac{1}{1-z} Z_{11} \hat{\mathcal{L}}_s \quad (s = 0, 1, 2, \dots). \tag{30}$$

Since the term $\frac{1}{z}[Z_1, \hat{\mathcal{L}}_s]$ is holomorphic at $z = 0$, $\hat{\mathcal{L}}_{s+1}(z)$ is uniquely determined as

$$\hat{\mathcal{L}}_{s+1}(z) = \int_0^z \left(\frac{1}{z}[Z_1, \hat{\mathcal{L}}_s] + \frac{1}{1-z} Z_{11} \hat{\mathcal{L}}_s \right) dz. \tag{31}$$

In terms of iterated integral, it is expressed as

$$\begin{aligned} \hat{\mathcal{L}}_s(z) = \sum_{k_1+\cdots+k_r=s} & \left\{ \int_0^z \zeta_1^{k_1-1} \circ \zeta_{11} \circ \cdots \circ \zeta_1^{k_r-1} \circ \zeta_{11} \right\} \\ & \times \operatorname{ad}(Z_1)^{k_1-1} \mu(Z_{11}) \cdots \operatorname{ad}(Z_1)^{k_r-1} \mu(Z_{11})(\mathbf{I}). \end{aligned} \tag{32}$$

Here $\operatorname{ad}(Z_1) \in \operatorname{End}(\mathcal{U}(\mathfrak{X}))$ stands for the adjoint operator by Z_1, and $\mu(Z_{11}) \in \operatorname{End}(\mathcal{U}(\mathfrak{X}))$ the multiplication of Z_{11} from the left. From these considerations, it follows that the fundamental solution normalized at $z = 0$ exists and is unique.

From (32), it follows that $\hat{\mathcal{L}}(z)$ is a grouplike element in $\widetilde{\mathcal{U}}(\mathfrak{X})$: Let $\Delta : \mathcal{U}(\mathfrak{X}) \longrightarrow \mathcal{U}(\mathfrak{X}) \otimes \mathcal{U}(\mathfrak{X})$ be the coproduct of $\mathcal{U}(\mathfrak{X})$. Then one can easily show that

$$\Delta(\hat{\mathcal{L}}_s(z)) = \sum_{s_1+s_2=s} \hat{\mathcal{L}}_{s_1}(z) \otimes \hat{\mathcal{L}}_{s_2}(z),$$

which is equivalent to

$$\Delta(\hat{\mathcal{L}}(z)) = \hat{\mathcal{L}}(z) \otimes \hat{\mathcal{L}}(z).$$

Hence $\hat{\mathcal{L}}(z)$ is grouplike and so is $\mathcal{L}(z)$.

The iterated integral in (32) is nothing but a multiple polylogarithm of one variable (MPL, for short):

$$\operatorname{Li}_{k_1,\dots,k_r}(z) = \int_0^z \zeta_1^{k_1-1} \circ \zeta_{11} \circ \cdots \circ \zeta_1^{k_r-1} \circ \zeta_{11}. \tag{33}$$

If $|z| < 1$, it has a Taylor expansion

$$\operatorname{Li}_{k_1,\dots,k_r}(z) = \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{z^{n_1}}{n_1^{k_1} \cdots n_r^{k_r}}. \tag{34}$$

If $k_1 \geq 2$, we have

$$\lim_{z\to 1-0} \mathrm{Li}_{k_1,\ldots,k_r}(z) = \zeta(k_1,\ldots,k_r), \tag{35}$$

where the right side above is the multiple zeta value:

$$\zeta(k_1,\ldots,k_r) = \sum_{n_1>\cdots>n_r>0} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}}. \tag{36}$$

3.2 The formal generalized 1KZ equation

Let us consider the following differential equation which is a generalization of the formal 1KZ equation. For mutually distinct points $a_1,\ldots,a_m \in \mathbf{C}-\{0\}$ we set

$$dG = \Omega G, \quad \Omega = \frac{dz}{z}X_0 + \sum_{i=1}^{m} \frac{a_i dz}{1-a_i z} X_i. \tag{37}$$

Here the coefficients $X_0, X_1, \ldots, X_m$ are free formal elements. For $r=1, a_1=1$, this is the formal 1KZ equation. This is a differential equation of Schlesinger type with regular singular points $0, 1/a_1, \ldots, 1/a_m, \infty$. We call (37) the formal generalized 1KZ equation.

Let $\mathfrak{X} = \mathbf{C}\{X_0, X_1, \ldots, X_m\}$ be the free Lie algebra generated by $X_0, X_1, \ldots, X_m$. As in the case of 1KZ, $\mathfrak{X}$ is a graded Lie algebra under the grading with respect to the degree of Lie polynomials. The universal enveloping algebra $\mathcal{U}(\mathfrak{X})$ is also a graded algebra with respect to the length of words, and one can define the completion $\widetilde{\mathcal{U}}(\mathfrak{X})$ of $\mathcal{U}(\mathfrak{X})$. A solution of this equation is a function which takes value in $\widetilde{\mathcal{U}}(\mathfrak{X})$.

The dual Hopf algebra of $\mathcal{U}(\mathfrak{X})$ is a free shuffle algebra $S(A)$ generated by the alphabet $A = \{\xi_0, \xi_1, \ldots, \xi_m\}$. Identifying

$$\xi_0 = \frac{dz}{z}, \qquad \xi_i = \frac{a_i dz}{1-a_i z}, \quad (1 \leq i \leq m) \tag{38}$$

one can define iterated integral for an element in $S(A)$. In that case, an element in $S(A)$ will be called an iterated form. The definition of iterated integral for an iterated form is just the same as in the 1KZ case.

The fundamental solution normalized at the origin $z=0$ of this equation, which we denote by $\mathcal{L}(z)$, is a solution satisfying the following condition:

$$\mathcal{L}(z) = \hat{\mathcal{L}}(z) z^{X_0} \tag{39}$$

where $\hat{\mathcal{L}}(z)$ is a function which takes the values in $\widetilde{\mathcal{U}}(\mathfrak{X})$, and holomorphic at $z=0$, and satisfies $\hat{\mathcal{L}}(0) = \mathbf{I}$. One can show the existence and the uniqueness of the solution, and that it is a grouplike element of $\widetilde{\mathcal{U}}(\mathfrak{X})$, in a completely parallel way: It is represented as $\hat{\mathcal{L}}(z) = \sum_{s=0}^{\infty} \hat{\mathcal{L}}_s(z)$, $\hat{\mathcal{L}}_0(z) = \mathbf{I}$, and

$$\begin{aligned}\hat{\mathcal{L}}_s(z) = \sum_{\substack{k_1+\cdots+k_r=s \\ i_1,\ldots,i_r\in\{1,\ldots,m\}}} & L({}^{k_1}a_{i_1}\cdots {}^{k_r}a_{i_r}; z) \\ & \times \mathrm{ad}(X_0)^{k_1-1}\mu(X_{i_1})\cdots \mathrm{ad}(X_0)^{k_r-1}\mu(X_{i_r})(\mathbf{I}).\end{aligned} \tag{40}$$

Here $\mathrm{ad}(X_0)$, $\mu(X_i)$ are defined in the same manner as in the 1KZ case, and $L({}^{k_1}a_{i_1}\cdots{}^{k_r}a_{i_r};z)$ denotes a hyperlogarithm of general type,

$$L({}^{k_1}a_{i_1}\cdots{}^{k_r}a_{i_r};z) := \int_0^z \xi_0^{k_1-1}\circ\xi_{i_1}\circ\xi_0^{k_2-1}\circ\xi_{i_2}\circ\cdots\circ\xi_0^{k_r-1}\circ\xi_{i_r}. \tag{41}$$

This is a many valued analytic function on $\mathbf{P}^1 - \{0, \frac{1}{a_{i_1}}, \cdots, \frac{1}{a_{i_r}}, \infty\}$. If $|z| < \min\{\frac{1}{|a_{i_1}|}, \ldots, \frac{1}{|a_{i_r}|}\}$, it has a Taylor expansion

$$L({}^{k_1}a_{i_1}\cdots{}^{k_r}a_{i_r};z) = \sum_{n_1>n_2>\cdots>n_r>0} \frac{a_{i_1}^{n_1-n_2}a_{i_2}^{n_2-n_3}\cdots a_{i_r}^{n_r}}{n_1^{k_1}\cdots n_r^{k_r}} z^{n_1}. \tag{42}$$

For $r=1$ and $a_1=1$, this is nothing but MPL. The hyperlogarithms of type $\mathcal{M}_{0,5}$ that we need in what follows are those of $m=2, a_1=1$.

4 The formal KZ equation of two variables

In this section, we consider the formal KZ equation of two variables in the cubic coordinate system, and the structure of the IPB Lie algebra for this equation. The tensor decomposition theorem for the universal enveloping algebra of the IPB Lie algebra will be proved.

4.1 The formal KZ equation of two variables in the cubic coordinate system

For $n=5$, the formal KZ equation in the cubic coordinate system (13) reads as follows:

$$dG = \Omega G, \quad \Omega = \frac{dz_1}{z_1}Z_1 + \frac{dz_1}{1-z_1}Z_{11} + \frac{dz_2}{z_2}Z_2 + \frac{dz_2}{1-z_2}Z_{22} + \frac{d(z_1z_2)}{1-z_1z_2}Z_{12}, \tag{43}$$

where $Z_1 = X_{12}+X_{13}+X_{23},\ Z_2 = X_{23},\ Z_{11} = -X_{14},\ Z_{22} = -X_{12},\ Z_{12} = -X_{24}$. This will be referred to as the formal 2KZ equation hereafter. The IPBR for this equation are

$$[Z_1, Z_2] = [Z_{11}, Z_2] = [Z_1, Z_{22}] = 0, \tag{44}$$

$$[Z_{11}, Z_{22}] = -[Z_{11}, Z_{12}] = [Z_{22}, Z_{12}] = -[Z_1 - Z_2, Z_{12}]. \tag{45}$$

The IPB Lie algebra $\mathfrak{X}$ is defined as a quotient of the free Lie algebra $\mathfrak{L} = \mathbf{C}\{Z_1, Z_{11}, Z_2, Z_{22}, Z_{12}\}$ by the ideal generated by the IPBR (44), (45). As discussed in the previous section, $\mathfrak{L}$ and the universal enveloping algebra $\mathcal{U}(\mathfrak{L})$ has grading with respect to the length of words. Since the IPBR is homogeneous, $\mathfrak{X}$ and the universal enveloping algebra $\mathcal{U}(\mathfrak{X})$ inherit the grading of $\mathfrak{L}, \mathcal{U}(\mathfrak{L})$:

$$\mathfrak{X} = \bigoplus_{s=1}^{\infty}\mathfrak{X}_s, \qquad [\mathfrak{X}_s, \mathfrak{X}_{s'}] \subset \mathfrak{X}_{s+s'}, \tag{46}$$

$$\mathcal{U}(\mathfrak{X}) = \bigoplus_{s=0}^{\infty}\mathcal{U}_s(\mathfrak{X}), \qquad \mathcal{U}_s(\mathfrak{X})\cdot\mathcal{U}_{s'}(\mathfrak{X}) \subset \mathcal{U}_{s+s'}(\mathfrak{X}). \tag{47}$$

Put

$$\zeta_i = \frac{dz_i}{z_i} \quad (i=1,2), \quad \zeta_{ii} = \frac{dz_i}{1-z_i} \quad (i=1,2), \quad \zeta_{12} = \frac{d(z_1z_2)}{1-z_1z_2}. \tag{48}$$

The nontrivial AR are

$$\zeta_{22}\wedge\zeta_{11} + \zeta_{11}\wedge\zeta_{12} + \zeta_{12}\wedge(\zeta_{22}+\zeta_2) = 0, \tag{49}$$

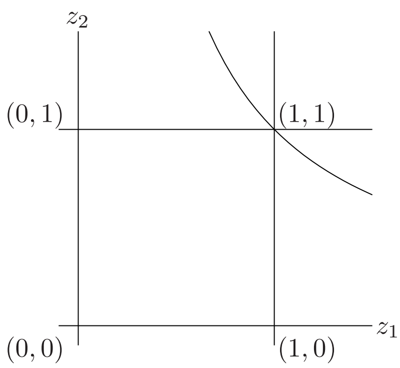
$$(\zeta_1+\zeta_2)\wedge\zeta_{12} = 0. \tag{50}$$

In terms of ζ_i, ζ_{ij}, Ω is expressed as

$$\Omega = \zeta_1 Z_1 + \zeta_{11}Z_{11} + \zeta_2 Z_2 + \zeta_{22}Z_{22} + \zeta_{12}Z_{12} \in \Omega^1_{\mathbf{X}}(\log D)\otimes\mathcal{U}_1(\mathfrak{X}) \tag{51}$$

where $\Omega^1_{\mathbf{X}}(\log D)$ stands for the space of differential one-forms on $\mathbf{X} = \mathbf{P}^1\times\mathbf{P}^1$ with logarithmic poles along the divisors $D = \{z_1 = 0,1,\infty\}\cup\{z_2 = 0,1,\infty\}\cup\{z_1z_2 = 1\}$. From IPBR (44), (45) and AR (49), (50), it follows that $\Omega\wedge\Omega = 0$. Hence the formal 2KZ equation is an integrable system on $\mathbf{X}$. A solution of the formal 2KZ equation is a many valued analytic function on $\mathbf{X} - D$ with values in $\widetilde{\mathcal{U}}(\mathfrak{X})$ (= the completion of $\mathcal{U}(\mathfrak{X})$ with respect to the above grading).

Note that the divisors D are normal crossing at $(z_1,z_2) = (0,0),(1,0),(0,1)$.



4.2 The tensor product decomposition theorem for $\mathcal{U}(\mathfrak{X})$

Next we will consider the tensor product decomposition theorem for $\mathcal{U}(\mathfrak{X})$.

Let $\mathfrak{X}^{(1)}_{1\otimes 2}$ (resp. $\mathfrak{X}^{(2)}_{1\otimes 2}$) be a subalgebra of $\mathfrak{X}$ generated by $\{Z_1, Z_{11}, Z_{12}\}$ (resp. $\{Z_2, Z_{22}\}$). Since there exists no relation among $\{Z_1, Z_{11}, Z_{12}\}$, (resp.$\{Z_2, Z_{22}\}$), they are free Lie algebras. Similarly set $\mathfrak{X}^{(2)}_{2\otimes 1}$ (resp. $\mathfrak{X}^{(1)}_{2\otimes 1}$) as the subalgebra generated by $\{Z_2, Z_{22}, Z_{12}\}$ (resp. $\{Z_1, Z_{11}\}$). They are also free Lie algebras:

$$\mathfrak{X}^{(i_1)}_{i_1\otimes i_2} = \mathbf{C}\{Z_{i_1}, Z_{i_1i_1}, Z_{12}\}, \quad \mathfrak{X}^{(i_2)}_{i_1\otimes i_2} = \mathbf{C}\{Z_{i_2}, Z_{i_2i_2}\} \quad (\{i_1,i_2\} = \{1,2\}). \tag{52}$$

For each $\mathfrak{A} = \mathfrak{X}$, $\mathfrak{X}^{(i_1)}_{i_1\otimes i_2}$, $\mathfrak{X}^{(i_2)}_{i_1\otimes i_2}$, we denote by $\mathcal{U}(\mathfrak{A})$ the universal enveloping algebra of $\mathfrak{A}$. They are graded algebras;

$$\mathcal{U}(\mathfrak{A}) = \bigoplus_{s=0}^{\infty}\mathcal{U}_s(\mathfrak{A}), \qquad \mathcal{U}_s(\mathfrak{A}) = \mathcal{U}(\mathfrak{A})\cap\mathcal{U}_s(\mathfrak{X}).$$

We also denote by $\mathcal{W}(\mathfrak{A})$ the set of all words of $\mathcal{U}(\mathfrak{A})$, by $\mathcal{W}^0(\mathfrak{A})$ the set of all words which do not end with Z_1, Z_2. Put $\mathcal{W}_s(\mathfrak{A}) = \mathcal{W}(\mathfrak{A}) \cap \mathcal{U}_s(\mathfrak{A})$ and $\mathcal{W}_s^0(\mathfrak{A}) = \mathcal{W}^0(\mathfrak{A}) \cap \mathcal{U}_s(\mathfrak{A})$. Then we have the following proposition:

Proposition 2 (Tensor product decomposition for $\mathcal{U}(\mathfrak{X})$). (i) The Lie algebras (52) satisfy

$$[\mathfrak{X}^{(i_1)}_{i_1\otimes i_2}, \mathfrak{X}^{(i_2)}_{i_1\otimes i_2}] \subset \mathfrak{X}^{(i_1)}_{i_1\otimes i_2} \quad (\{i_1, i_2\} = \{1, 2\}). \tag{53}$$

(ii) As vector spaces, we have, for $\{i_1, i_2\} = \{1, 2\}$,

$$\mathfrak{X} = \mathfrak{X}^{(i_1)}_{i_1\otimes i_2} \oplus \mathfrak{X}^{(i_2)}_{i_1\otimes i_2}, \tag{54}$$

$$\mathcal{U}(\mathfrak{X}) = \mathcal{U}(\mathfrak{X}^{(i_1)}_{i_1\otimes i_2}) \otimes \mathcal{U}(\mathfrak{X}^{(i_2)}_{i_1\otimes i_2}). \tag{55}$$

In particular, the sets $\{W_1W_2 \mid W_1 \in \mathcal{W}(\mathfrak{X}^{(i_1)}_{i_1\otimes i_2}),\ W_2 \in \mathcal{W}(\mathfrak{X}^{(i_2)}_{i_1\otimes i_2})\}$ $(\{i_1, i_2\} = \{1, 2\})$ give a basis of $\mathcal{U}(\mathfrak{X})$, respectively.

Proof. (i) From (44), we have

$$[Z_1, Z_2] = [Z_{11}, Z_2] = 0 \in \mathfrak{X}^{(1)}_{1\otimes 2},$$

and from (45)

$$[Z_{12}, Z_2] = [Z_{12}, Z_1] - [Z_{12}, Z_{11}] \in \mathfrak{X}^{(1)}_{1\otimes 2}.$$

For $A_1, A_2 \in \mathfrak{X}^{(1)}_{1\otimes 2}$, we have

$$[Z_2, [A_1, A_2]] = [A_1, [Z_2, A_2]] + [[Z_2, A_1], A_2].$$

Hence one can show that $[\mathfrak{X}^{(1)}_{1\otimes 2}, Z_2] \subset \mathfrak{X}^{(1)}_{1\otimes 2}$ by induction on the degree of elements in $\mathfrak{X}^{(1)}_{1\otimes 2}$.

Next, form (44), $[Z_1, Z_{22}] = 0$, and from (45)

$$[Z_{11}, Z_{22}] = [Z_{12}, Z_{11}] \in \mathfrak{X}^{(1)}_{1\otimes 2}, \quad [Z_{12}, Z_{22}] = -[Z_{12}, Z_{11}] \in \mathfrak{X}^{(1)}_{1\otimes 2},$$

so that

$$[\mathfrak{X}^{(1)}_{1\otimes 2}, Z_{22}] \subset \mathfrak{X}^{(1)}_{1\otimes 2}.$$

By induction on the degree of elements in $\mathfrak{X}^{(1)}_{1\otimes 2}$, one can show that

$$[\mathfrak{X}^{(1)}_{1\otimes 2}, \mathfrak{X}^{(2)}_{1\otimes 2}] \subset \mathfrak{X}^{(1)}_{1\otimes 2}.$$

Similarly one can prove $[\mathfrak{X}^{(2)}_{2\otimes 1}, \mathfrak{X}^{(1)}_{2\otimes 1}] \subset \mathfrak{X}^{(2)}_{2\otimes 1}$.

(ii) This is clear from (i) and the standard argument of universal enveloping algebras. ☐

5 The reduced bar algebra

Let $\mathcal{U}(\mathfrak{X})$ be the universal enveloping algebra discussed in the previous section. In this section, we introduce an algebra of differential forms which is a dual Hopf algebra of $\mathcal{U}(\mathfrak{X})$. Since $\mathcal{U}(\mathfrak{X})$ is a quotient algebra of the non-commutative

polynomial algebra, its dual is a subalgebra of the free shuffle algebra. In fact, it is an algebra consisting of differential forms satisfying Chen's integrability condition. Iterated integrals associated with such forms depends on only the homotopy class of the integration contours.

The algebra will be referred to as the reduced bar algebra. It coincides with the 0-th cohomology of the reduced bar complex [C1],[Ha],[B] of the Orlik-Solomon algebra associated with the hyperplanes arrangement associated with the Dynkin diagram of type A_4 [OS], [OT].

5.1 Chen's integrability condition and the reduced bar algebra

Let $A=\{\zeta_1,\zeta_{11},\zeta_2,\zeta_{22},\zeta_{12}\}$ be an alphabet and $S(A)=(\mathbf{C}\langle A\rangle,\sqcup\!\sqcup)$ be a free shuffle algebra in A. We denote by $\circ$ the usual product by concatenation of $S(A)$ and $\sqcup\!\sqcup$ the shuffle product of $S(A)$. $S(A)=\bigoplus_{s=0}^{\infty}S_s(A)$ is a graded $\sqcup\!\sqcup$-algebra with respect to the length of words. Let Δ^* be a coproduct, ε^* a counit and S^* an antipode of $S(A)$ defined by

$$\Delta^*(\omega_{i_1}\circ\cdots\circ\omega_{i_s})=\sum_{l=0}^{s}(\omega_{i_1}\circ\cdots\circ\omega_{i_l})\otimes(\omega_{i_{l+1}}\circ\cdots\circ\omega_{i_s}), \tag{56}$$

$$\varepsilon^*(\omega_{i_1}\circ\cdots\circ\omega_{i_s})=0, \tag{57}$$

$$S^*(\omega_{i_1}\circ\cdots\circ\omega_{i_s})=(-1)^s(\omega_{i_s}\circ\cdots\circ\omega_{i_1}). \tag{58}$$

where $\omega_i\in A$. Then $(S(A),\sqcup\!\sqcup,\mathbf{1},\Delta^*,\varepsilon^*,S^*)$ is a commutative graded Hopf algebra.

Let $\mathfrak{L}=\mathbf{C}\{Z_1,Z_{11},Z_2,Z_{22},Z_{12}\}$ be a free Lie algebra and $\mathfrak{H}=\mathcal{U}(\mathfrak{L})$ the universal enveloping algebra of $\mathfrak{L}$. They are graded (Lie) algebras: $\mathfrak{L}=\oplus_{s=1}^{\infty}\mathfrak{L}_s$, $\mathfrak{H}=\oplus_{s=0}^{\infty}\mathfrak{H}_s$. $\mathfrak{H}$ is a restricted dual Hopf algebra of $S(A)$ by identifying the dual elements of $(Z_1,Z_{11},Z_2,Z_{22},Z_{12})$ with $(\zeta_1,\zeta_{11},\zeta_2,\zeta_{22},\zeta_{12})$.

Identifying the letters $\zeta_1,\ldots,\zeta_{12}$ with differential forms through (48), we will call an element of $S(A)$ an iterated form.

Definition 3 (Chen's integrability condition)**.** We say that an iterated form

$$\varphi=\sum_{I=\{i_1,\ldots,i_s\}}c_I\,\omega_{i_1}\circ\cdots\circ\omega_{i_s}\quad\in S_s(A)$$

(where each $\omega_{i_\alpha}\in A,\ \ C_I\in\mathbf{C}$) satisfies Chen's integrability condition if, for all l $(1\le l<s)$,

$$\sum_I c_I\,\omega_{i_1}\otimes\cdots\otimes\omega_{i_l}\wedge\omega_{i_{l+1}}\otimes\cdots\otimes\omega_{i_s}=0 \tag{59}$$

holds as a multiple differential form.

Let $\mathcal{B}$ be a subspace of $S(A)$ spanned by elements satisfying Chen's integrability condition. $\mathcal{B}$ also have the grading $\mathcal{B}=\bigoplus_{s=0}^{\infty}\mathcal{B}_s,\quad\mathcal{B}_s=\mathcal{B}\cap S_s(A)$ as vector space. We obtain $\mathcal{B}_0=\mathbf{C}\mathbf{1},\ \mathcal{B}_1=\mathbf{C}\zeta_1\oplus\mathbf{C}\zeta_{11}\oplus\mathbf{C}\zeta_2\oplus\mathbf{C}\zeta_{22}\oplus\mathbf{C}\zeta_{12}$

clearly and $\mathcal{B}_2$ is a 19 dimensional vector space given by

$$\begin{aligned}\mathcal{B}_2 = & \bigoplus_{\omega\in A} \mathbf{C}\omega\circ\omega \oplus \bigoplus_{i=1,2} \mathbf{C}\zeta_i\circ\zeta_{ii} \oplus \bigoplus_{i=1,2} \mathbf{C}\zeta_{ii}\circ\zeta_i \\ & \oplus \bigoplus_{\substack{\omega_1=\zeta_1,\zeta_{11}\\ \omega_2=\zeta_2,\zeta_{22}}} \mathbf{C}(\omega_1\circ\omega_2+\omega_2\circ\omega_1) \oplus \bigoplus_{\omega\in A-\{\zeta_{12}\}} \mathbf{C}(\omega\circ\zeta_{12}+\zeta_{12}\circ\omega) \\ & \oplus \mathbf{C}(\zeta_1\circ\zeta_{12}+\zeta_2\circ\zeta_{12}) \oplus \mathbf{C}(\zeta_{11}\circ\zeta_{12}+\zeta_{22}\circ\zeta_{11}-\zeta_{22}\circ\zeta_{12}-\zeta_2\circ\zeta_{12}) \end{aligned} \tag{60}$$

by the AR (49) and (50). Furthermore we have

$$\mathcal{B}_s = \bigcap_{j=1}^{s-1} \mathcal{B}_j\circ\mathcal{B}_{s-j} = \bigcap_{j=0}^{s-2} \underbrace{\mathcal{B}_1\circ\cdots\circ\mathcal{B}_1}_{j \text{ times}} \circ\mathcal{B}_2\circ \underbrace{\mathcal{B}_1\circ\cdots\circ\mathcal{B}_1}_{s-j-2 \text{ times}} \tag{61}$$

for $s>2$ [B]. $(\mathcal{B}, ш, \mathbf{1}, \Delta^*, \varepsilon^*, S^*)$ is a graded Hopf subalgebra of $S(A)$ and $\mathcal{B}$ is referred to the reduced bar algebra. In this notation, Ω (51) belongs to $\mathcal{B}_1\otimes\mathcal{U}_1(\mathfrak{X})$.

Now let $\mathfrak{I}_2 = \mathbf{C}[Z_1,Z_2]+\mathbf{C}[Z_1,Z_{22}]+\mathbf{C}[Z_{11},Z_2]+\mathbf{C}([Z_{11},Z_{22}]+[Z_{11},Z_{12}])+\mathbf{C}([Z_{11},Z_{22}]+[Z_{12},Z_{22}])+\mathbf{C}([Z_{11},Z_{22}]+[Z_1-Z_2,Z_{12}])$ be a subspace of $\mathfrak{L}_2$ and $\mathfrak{I}$ (resp. $\mathfrak{J}$) be an ideal of $\mathfrak{L}$ (resp. a two sided ideal of $\mathfrak{H}$) generated by $\mathfrak{I}_2$. Then $\mathfrak{X}=\mathfrak{L}/\mathfrak{I}$ and $\mathcal{U}(\mathfrak{X})=\mathfrak{H}/\mathfrak{J}$.

Proposition 4 ([B])**.** $\mathcal{U}(\mathfrak{X})$ is a graded restricted dual Hopf algebra of $\mathcal{B}$.

For the proof we need the following lemma.

Lemma 5.

$$(\underbrace{\mathcal{B}_1\circ\cdots\circ\mathcal{B}_1}_{l-1 \text{ times}}\circ\mathcal{B}_2\circ\underbrace{\mathcal{B}_1\circ\cdots\circ\mathcal{B}_1}_{s-l-1 \text{ times}})^{\perp}\cap\mathfrak{H}_s = \mathfrak{H}_{l-1}\mathfrak{I}_2\mathfrak{H}_{s-l-1}$$

Proof. Let

$$\{\zeta_{i_1}\circ\cdots\circ\zeta_{i_{l-1}}\circ\varphi\circ\zeta_{i_{l+1}}\circ\cdots\circ\zeta_{i_s} \mid i_k\in\{1,11,2,22,12\},\ \varphi\in \text{the basis of } \mathcal{B}_2\}$$

be a basis of $\mathcal{B}_1\circ\cdots\circ\mathcal{B}_1\circ\mathcal{B}_2\circ\mathcal{B}_1\circ\cdots\circ\mathcal{B}_1$. An element of $\mathfrak{H}_s$ which is orthogonal to $\zeta_{i_1}\circ\cdots\circ\zeta_{i_{l-1}}\circ\varphi\circ\zeta_{i_{l+1}}\circ\cdots\circ\zeta_{i_s}$ can be expressed as $Z_{i_1}\circ\cdots\circ Z_{i_{l-1}}\circ\Phi\circ Z_{i_{l+1}}\circ\cdots\circ Z_{i_s}$, for some $\Phi\in\mathfrak{H}_2$ such that $<\Phi,\varphi>=0$. On the other hand, we obtain $\mathcal{B}_2^{\perp}\cap\mathfrak{H}_2=\mathfrak{I}_2$ by counting the dimensions. We have thus proved the lemma. □

Proof of Proposition 4. Since $\mathfrak{J}_s=\sum_{l=1}^{s-1}\mathfrak{H}_{l-1}\mathfrak{I}_2\mathfrak{H}_{s-l-1}$ $(\mathfrak{J}_2=\mathfrak{I}_2)$, the lemma tells that $\mathfrak{J}$ is the orthogonal complement of $\mathcal{B}$. Hence $\mathcal{U}(\mathfrak{X})=\mathfrak{H}/\mathfrak{J}$ is a dual Hopf algebra of $\mathcal{B}$. □

5.2 Iterated integral of two variable

For an iterated form

$$\varphi = \sum_{I=\{i_1,\ldots,i_s\}} c_I\omega_{i_1}\circ\cdots\circ\omega_{i_s}\in\mathcal{B}_s \qquad (\omega_{i_k}\in\mathcal{B}_1),$$

we define iterated integral $\int \varphi$ as follows: From $\mathcal{B}_s \subset \mathcal{B}_1 \circ \mathcal{B}_{s-1}$, we can write

$$\varphi = \sum_{i_1} \omega_{i_1} \circ \Big(\sum_{I'=\{i_2,\dots,i_s\}} c_{i_1,I'}\, \omega_{i_2} \circ \cdots \circ \omega_{i_s} \Big),$$

as $\omega_{i_1} \in \mathcal{B}_1$, $\sum_{I'} c_{i_1,I'}\, \omega_{i_2} \circ \cdots \circ \omega_{i_s} \in \mathcal{B}_{s-1}$. Then we define

$$\begin{cases} \displaystyle\int_{(z_1^{(0)},z_2^{(0)})}^{(z_1,z_2)} \mathbf{1} \; = 1 \;\text{ (constant function)}, \\ \displaystyle\int_{(z_1^{(0)},z_2^{(0)})}^{(z_1,z_2)} \varphi \; = \int_{(z_1^{(0)},z_2^{(0)})}^{(z_1,z_2)} \sum_{i_1} \omega_{i_1}(z_1', z_2') \int_{(z_1^{(0)},z_2^{(0)})}^{(z_1',z_2')} \sum_{I'=\{i_2,\dots,i_s\}} c_{i_1,I'}\, \omega_{i_2} \circ \cdots \circ \omega_{i_s}, \end{cases} \tag{62}$$

where $\omega_{i_1}(z_1', z_2')$ stands for the form of dz_1', dz_2' defined by displacing z_1, z_2 to z_1', z_2' in ω_{i_1}. The iterated integral of $\varphi \in \mathcal{B}$ yields a many-valued analytic function on $\mathbf{X} - D$ by the lemma due to Chen.

Lemma 6 (Chen's lemma[C2])**.** For any iterated form $\varphi \in \mathcal{B}$, the value of the iterated integral of φ depends on only the homotopy class of the integral contour.

Moreover, in a similar fashion of one variable, the iterated integral is $ш$-homomorphism namely

$$\int \varphi_1 ш \varphi_2 = \Big(\int \varphi_1 \Big) \Big(\int \varphi_2 \Big)$$

for all $\varphi_1, \varphi_2 \in \mathcal{B}$.

5.3 The tensor product decomposition of $\mathcal{B}$ and two specific contours $C_{1\otimes 2}, C_{2\otimes 1}$

We denote by $\mathcal{B}^{(1)}_{1\otimes 2}$ and $\mathcal{B}^{(2)}_{1\otimes 2}$ (resp. $\mathcal{B}^{(2)}_{2\otimes 1}$ and $\mathcal{B}^{(1)}_{2\otimes 1}$) the dual Hopf algebra of $\mathcal{U}(\mathfrak{X}^{(1)}_{1\otimes 2})$ and $\mathcal{U}(\mathfrak{X}^{(2)}_{1\otimes 2})$ (resp. $\mathcal{U}(\mathfrak{X}^{(2)}_{2\otimes 1})$ and $\mathcal{U}(\mathfrak{X}^{(1)}_{2\otimes 1})$). By virtue of Proposition 2, we have already $\mathcal{U}(\mathfrak{X}) = \mathcal{U}(\mathfrak{X}^{(1)}_{1\otimes 2}) \otimes \mathcal{U}(\mathfrak{X}^{(2)}_{1\otimes 2}) = \mathcal{U}(\mathfrak{X}^{(2)}_{2\otimes 1}) \otimes \mathcal{U}(\mathfrak{X}^{(1)}_{2\otimes 1})$. Furthermore Proposition 4 says that $\mathcal{U}(\mathfrak{X})$ and $\mathcal{B}$ are dual to each other. Therefore $\mathcal{B}$ satisfies the tensor decomposition through the duality

$$\mathcal{B} \;\cong\; \mathcal{B}^{(1)}_{1\otimes 2} \otimes \mathcal{B}^{(2)}_{1\otimes 2} \;\cong\; \mathcal{B}^{(2)}_{2\otimes 1} \otimes \mathcal{B}^{(1)}_{2\otimes 1}. \tag{63}$$

On the other hand, since $\mathcal{U}(\mathfrak{X}^{(1)}_{1\otimes 2})$ and $\mathcal{U}(\mathfrak{X}^{(2)}_{1\otimes 2})$ (resp. $\mathcal{U}(\mathfrak{X}^{(2)}_{2\otimes 1})$ and $\mathcal{U}(\mathfrak{X}^{(1)}_{2\otimes 1})$) are universal enveloping algebra of a free Lie algebra, their dual Hopf algebras $\mathcal{B}^{(1)}_{1\otimes 2}$ and $\mathcal{B}^{(2)}_{1\otimes 2}$ (resp. $\mathcal{B}^{(2)}_{2\otimes 1}$ and $\mathcal{B}^{(1)}_{2\otimes 1}$) are isomorphic to the free shuffle algebras $S(A^{(1)}_{1\otimes 2})$ and $S(A^{(2)}_{1\otimes 2})$ (resp. $S(A^{(2)}_{2\otimes 1})$ and $S(A^{(1)}_{2\otimes 1})$). Where $S(A^{(1)}_{1\otimes 2})$ and $S(A^{(2)}_{1\otimes 2})$ (resp. $S(A^{(2)}_{2\otimes 1})$ and $S(A^{(1)}_{2\otimes 1})$) are free shuffle algebras generated by alphabets $A^{(1)}_{1\otimes 2} = \{\zeta_1, \zeta_{11}, \zeta^{(1)}_{12}\}$ and $A^{(2)}_{1\otimes 2} = \{\zeta_2, \zeta_{22}\}$ (resp. $A^{(2)}_{2\otimes 1} = \{\zeta_2, \zeta_{22}, \zeta^{(2)}_{12}\}$ and $A^{(1)}_{2\otimes 1} = \{\zeta_1, \zeta_{11}\}$). We identify the letter $\zeta^{(i)}_{12}$ as a differential form

$$\zeta^{(i)}_{12} = \frac{z^{(i)}_{12} dz_i}{1 - z_{12}}$$

and note that $\zeta_{12} = \zeta_{12}^{(1)} + \zeta_{12}^{(2)}$. We have the following lemma.

Lemma 7. There is a natural isomorphism preserving the gradings as a vector space,

$$\mathcal{B} \cong S(A_{1\otimes 2}^{(1)}) \otimes S(A_{1\otimes 2}^{(2)}) \cong S(A_{2\otimes 1}^{(2)}) \otimes S(A_{2\otimes 1}^{(1)}). \tag{64}$$

Next we define linear projections $\mathrm{Pr}_{i_1\otimes i_2}^{(i_k)} : S(A) \to S(A_{i_1\otimes i_2}^{(i_k)})$ ($\{i_1, i_2\} = \{1, 2\}$, $i_k \in \{1, 2\}$) as follows: For $i_k = i_1$, send dz_{i_2} to 0, and for $i_k = i_2$, send dz_{i_1} to 0.

Definition 8. We define linear maps $\iota_{i_1\otimes i_2} : \mathcal{B} \to S(A_{i_1\otimes i_2}^{(i_1)}) \otimes S(A_{i_1\otimes i_2}^{(i_2)})$ by

$$\iota_{i_1\otimes i_2} = \left(\mathrm{Pr}_{i_1\otimes i_2}^{(i_1)}\Big|_{\mathcal{B}} \otimes \mathrm{Pr}_{i_1\otimes i_2}^{(i_2)}\Big|_{\mathcal{B}} \right) \circ \Delta^*. \tag{65}$$

Namely $\iota_{1\otimes 2}$ (resp. $\iota_{2\otimes 1}$) is the projection picking up components which have the form that all $\zeta_1, \zeta_{11}, \zeta_{12}$ (resp. $\zeta_2, \zeta_{22}, \zeta_{12}$) appear in the left of all ζ_2, ζ_{22} (resp. ζ_1, ζ_{11}). Since $\mathrm{Pr}_{i_1\otimes i_2}^{(i_k)}$ and Δ^* are ш-algebra homomorphisms, so are $\iota_{1\otimes 2}$ and $\iota_{2\otimes 1}$.

Proposition 9. $\iota_{1\otimes 2} : \mathcal{B} \to S(A_{1\otimes 2}^{(1)}) \otimes S(A_{1\otimes 2}^{(2)})$, $\iota_{2\otimes 1} : \mathcal{B} \to S(A_{2\otimes 1}^{(2)}) \otimes S(A_{2\otimes 1}^{(1)})$ are both ш-algebra isomorphisms.

Proof. We prove that $\iota_{1\otimes 2}$ is an isomorphism. From Lemma 7, $\dim \mathcal{B}_s = \dim \left(S(A_{1\otimes 2}^{(1)}) \otimes S(A_{1\otimes 2}^{(2)}) \right)_s$ and $\iota_{1\otimes 2}$ preserves the grading. Hence it is enough to prove that $\iota_s := \iota_{1\otimes 2}|_{\mathcal{B}_s}$ is injective. We show $\ker(\iota_s) = 0$ by induction on s.

For $s = 0$ and 1, $\ker(\iota_s) = 0$ is clear. For $s = 2$, we can show by direct calculation. For $s \geq 3$, we assume $\ker(\iota_{s-1}) = 0$. Set

$$\begin{aligned} S_{1\otimes 2,s}(A) &:= \{\varphi \in S_s(A) \mid \text{In all the terms of } \varphi, \\ &\qquad \zeta_2 \text{ or } \zeta_{22} \text{ appears in the left side of some } \zeta_1, \zeta_{11}, \zeta_{12}\}, \\ S_s^c(\zeta_2, \zeta_{22}) &:= \{\varphi \in S_s(A) \mid \varphi \text{ is a linear combination of words} \\ &\qquad \text{which have at least one } \zeta_1, \zeta_{11}, \zeta_{12}\}. \end{aligned}$$

Then $\ker(\iota_s) = \mathcal{B}_s \cap S_{1\otimes 2,s}(A)$.

Put $\varphi \in \ker(\iota_s)$. Since $\varphi \in S_{1\otimes 2,s}(A)$, φ can be written as

$$\varphi = \zeta_1 \circ \varphi_1 + \zeta_{11} \circ \varphi_{11} + \zeta_{12} \circ \varphi_{12} + \zeta_2 \circ \varphi_2 + \zeta_{22} \circ \varphi_{22},$$

here $\varphi_1, \varphi_{11}, \varphi_{12} \in S_{1\otimes 2,s-1}(A)$, $\varphi_2, \varphi_{22} \in S_{s-1}^c(\zeta_2, \zeta_{22})$. On the other hand, from the assumption, $\varphi_1, \varphi_{11}, \varphi_{12} \in \mathcal{B}_{s-1} \cap S_{1\otimes 2,s-1}(A) = \ker(\iota_{s-1}) = 0$ and

$$\varphi = \zeta_2 \circ \varphi_2 + \zeta_{22} \circ \varphi_{22}.$$

From the following lemma, we have $\varphi = 0$, and the proof is completed. □

Lemma 10. Assume that $\varphi \in \mathcal{B}_s \cap (S_1(\zeta_2, \zeta_{22}) \circ S_{s-1}(A))$. Then $\varphi \in S_s(\zeta_2, \zeta_{22})$.

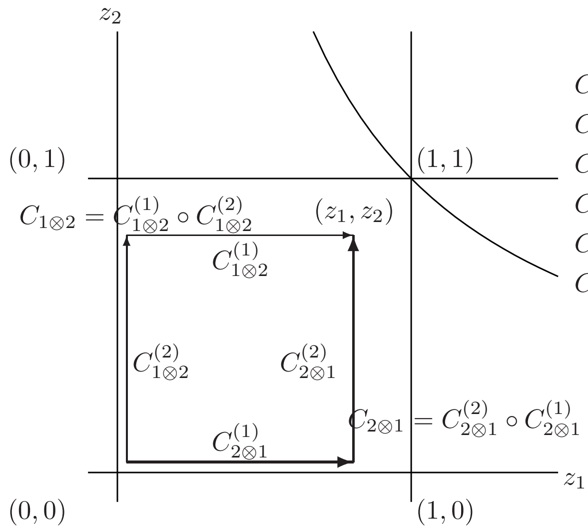
Proof. The lemma can be proved easily by direct calculation for $\mathcal{B}_2$ and by induction on s. □

We denote by $\mathcal{B}^0$ (resp. $S^0(A), S^0(A^{(1)}_{1\otimes 2}), S^0(A^{(2)}_{1\otimes 2}), S^0(A^{(2)}_{2\otimes 1}), S^0(A^{(1)}_{2\otimes 1})$) the subspace of $\mathcal{B}$ (resp. $S(A), S(A^{(1)}_{1\otimes 2}), S(A^{(2)}_{1\otimes 2}), S(A^{(2)}_{2\otimes 1}), S(A^{(1)}_{2\otimes 1})$) spanned by elements which do not have the component terminated by ζ_1 or ζ_2. Clearly $\mathcal{B}^0$ is a ш-subalgebra of $\mathcal{B}$. We note that $\int_{(0,0)}^{(z_1,z_2)} \varphi$ can be defined for $\varphi \in \mathcal{B}^0$. For $\mathcal{B}^0$ we have the following corollary.

Corollary 11. $\iota_{1\otimes 2} : \mathcal{B}^0 \to S^0(A^{(1)}_{1\otimes 2}) \otimes S^0(A^{(2)}_{1\otimes 2})$, $\iota_{2\otimes 1} : \mathcal{B}^0 \to S^0(A^{(2)}_{2\otimes 1}) \otimes S^0(A^{(1)}_{2\otimes 1})$ are both ш-algebra isomorphisms.

Proof. One can see easily $\iota_{i_1\otimes i_2}(B^0) \subset S^0(A^{(i_1)}_{i_1\otimes i_2}) \otimes S^0(A^{(i_2)}_{i_1\otimes i_2})$ by the definition of $\mathcal{B}$ and $\dim B^0_s = \dim \left(S^0(A^{(i_1)}_{i_1\otimes i_2}) \otimes S^0(A^{(i_2)}_{i_1\otimes i_2})\right)_s$ by the duality between $\mathcal{U}(\mathfrak{X})$ and $\mathcal{B}$. Hence the corollary holds. □

Assume that $0 < |z_1|, |z_2| < 1$ and consider the following two contours $C_{1\otimes 2},\ C_{2\otimes 1} : (0,0) \to (z_1, z_2)$.



$C_{1\otimes 2} = C^{(1)}_{1\otimes 2} \circ C^{(2)}_{1\otimes 2}$,
$C^{(2)}_{1\otimes 2} : (0,0) \to (0, z_2)$,
$C^{(1)}_{1\otimes 2} : (0, z_2) \to (z_1, z_2)$.
$C_{2\otimes 1} = C^{(2)}_{2\otimes 1} \circ C^{(1)}_{2\otimes 1}$,
$C^{(1)}_{2\otimes 1} : (0,0) \to (z_1, 0)$,
$C^{(2)}_{2\otimes 1} : (z_1, 0) \to (z_1, z_2)$.

Here the composition of paths $C \circ C'$ is defined by connecting C after C'.

For $\psi_1 \otimes \psi_2 \in S^0(A^{(1)}_{1\otimes 2}) \otimes S^0(A^{(2)}_{1\otimes 2})$, the integral $\int_{C_{1\otimes 2}} \psi_1 \otimes \psi_2$ is defined by

$$\int_{C_{1\otimes 2}} \psi_1 \otimes \psi_2 := \int_{z_1=0}^{z_1} \psi_1 \int_{z_2=0}^{z_2} \psi_2$$

and for $\psi_1 \otimes \psi_2 \in S^0(A^{(2)}_{2\otimes 1}) \otimes S^0(A^{(1)}_{2\otimes 1})$, the integral $\int_{C_{2\otimes 1}} \psi_1 \otimes \psi_2$ is defined by

$$\int_{C_{2\otimes 1}} \psi_1 \otimes \psi_2 := \int_{z_2=0}^{z_2} \psi_1 \int_{z_1=0}^{z_1} \psi_2.$$

Since the map $\iota_{1\otimes 2}$ (resp. $\iota_{2\otimes 1}$) picks up the terms of $\mathcal{B}^0$ whose iterated integral along $C_{1\otimes 2}$ (resp. $C_{2\otimes 1}$) does not vanish, we have

$$\int_{(0,0)}^{(z_1,z_2)} \varphi = \int_{(0,0)}^{(z_1,z_2)} \iota_{1\otimes 2}(\varphi) = \int_{C_{1\otimes 2}} \iota_{1\otimes 2}(\varphi) \tag{66}$$

$$= \int_{(0,0)}^{(z_1,z_2)} \iota_{2\otimes 1}(\varphi) = \int_{C_{2\otimes 1}} \iota_{2\otimes 1}(\varphi) \tag{67}$$

for $\varphi \in \mathcal{B}^0$.

5.4 Injectivity of the iterated integral of two variables

The iterated integral $\displaystyle\int_{(0,0)}^{(z_1,z_2)}$ defines an algebra homomorphism from $(\mathcal{B}^0, ш)$ to the algebra of all many valued analytic functions on $\mathbf{X} - D$. We denote by $\mathcal{F}$ the image of this homomorphism. By using the result of Brown([B], Theorem 3.26), we have the following claim.

Proposition 12. $\displaystyle\int_{(0,0)}^{(z_1,z_2)} : (\mathcal{B}^0, ш) \to \mathcal{F}$ is an algebra isomorphism.

From (66) and (67), we have

$$\int_{(0,0)}^{(z_1,z_2)} = \int_{C_{1\otimes 2}} \circ\, \iota_{1\otimes 2}, \qquad \int_{(0,0)}^{(z_1,z_2)} = \int_{C_{2\otimes 1}} \circ\, \iota_{2\otimes 1}. \tag{68}$$

By Corollary 11, $\iota_{1\otimes 2}, \iota_{2\otimes 1}$ are isomorphisms, so that we have the following.

Corollary 13. The integrals

$$\int_{C_{1\otimes 2}} : S^0(A^{(1)}_{1\otimes 2}) \otimes S^0(A^{(2)}_{1\otimes 2}) \to \mathcal{F}, \qquad \int_{C_{2\otimes 1}} : S^0(A^{(2)}_{2\otimes 1}) \otimes S^0(A^{(1)}_{2\otimes 1}) \to \mathcal{F}$$

are injective.

6 The normalized fundamental solution to the formal 2KZ equation

In this section, we construct the fundamental solution of the formal 2KZ equation normalized at the origin and prove the decomposition theorem for that.

6.1 Existence and uniqueness of the normalized fundamental solution

Let us define Ω_0 and Ω' by

$$\Omega_0 = \zeta_1 Z_1 + \zeta_2 Z_2, \quad \Omega' = \Omega - \Omega_0 = \zeta_{11} Z_{11} + \zeta_{22} Z_{22} + \zeta_{12} Z_{12}. \tag{69}$$

Namely Ω_0 is the singular part of Ω at $(0,0)$ and Ω' the regular part.

The fundamental solution normalized at $(0,0)$ of the formal 2KZ equation is, by definition,

$$\mathcal{L}(z_1,z_2)=\hat{\mathcal{L}}(z_1,z_2)z_1^{Z_1}z_2^{Z_2},\qquad \hat{\mathcal{L}}(z_1,z_2)=\sum_{s=0}^{\infty}\hat{\mathcal{L}}_s(z_1,z_2),\tag{70}$$

where $\hat{\mathcal{L}}_s(z_1,z_2)$ is a $\mathcal{U}_s(\mathfrak{X})$-valued many-valued analytic function holomorphic in a neighborhood of $(0,0)$ and $\hat{\mathcal{L}}_0(z_1,z_2)=\mathbf{I}$, $\hat{\mathcal{L}}_s(0,0)=0$ $(s>0)$.

From $[Z_1,Z_2]=[Z_1,Z_{22}]=[Z_{11},Z_2]=0$, $\hat{\mathcal{L}}(z_1,z_2)$ satisfies

$$d\hat{\mathcal{L}}(z_1,z_2)=[\Omega_0,\hat{\mathcal{L}}(z_1,z_2)]+\Omega'\hat{\mathcal{L}}(z_1,z_2).\tag{71}$$

This is equivalent to the recursive differential relations

$$d\hat{\mathcal{L}}_{s+1}(z_1,z_2)=[\Omega_0,\hat{\mathcal{L}}_s(z_1,z_2)]+\Omega'\hat{\mathcal{L}}_s(z_1,z_2)\qquad(s=0,1,2,\ldots).\tag{72}$$

Since $d\Omega_0=d\Omega'=0,\ \Omega_0\wedge\Omega_0=0,\ \Omega_0\wedge\Omega'+\Omega'\wedge\Omega_0+\Omega'\wedge\Omega'=0$, (71) is an integrable system. Note that, if $\hat{\mathcal{L}}_s(0,0)=0$, the right hand side of (71) is holomorphic at $(0,0)$ (See the discussion in Section 3.1), thus $\{\hat{\mathcal{L}}_s(z_1,z_2)\}$ should be represented by iterated integral as follows:

Proposition 14. There exists uniquely the fundamental solution of the formal 2KZ equation normalized at the origin: It is expressed as

$$\mathcal{L}(z_1,z_2)=\hat{\mathcal{L}}(z_1,z_2)z_1^{Z_1}z_2^{Z_2},\qquad \hat{\mathcal{L}}(z_1,z_2)=\sum_{s=0}^{\infty}\hat{\mathcal{L}}_s(z_1,z_2),$$
$$\hat{\mathcal{L}}_s(z_1,z_2)=\int_{(0,0)}^{(z_1,z_2)}\big(\mathrm{ad}(\Omega_0)+\mu(\Omega')\big)^s(\mathbf{1}\otimes\mathbf{I}),\tag{73}$$

where the actions ad and μ on $S(A)\otimes\mathcal{U}(\mathfrak{X})$ stand for

$$\mathrm{ad}(\omega\otimes X)(\varphi\otimes F)=(\omega\circ\varphi)\otimes[X,F],$$
$$\mu(\omega\otimes X)(\varphi\otimes F)=(\omega\circ\varphi)\otimes XF$$

for $\varphi\otimes F\in S(A)\otimes\mathcal{U}(\mathfrak{X}),\ \omega\otimes X\in\mathcal{B}_1\otimes\mathfrak{X}$.

Proposition 15. For each $s\geq 0$, $\big(\mathrm{ad}(\Omega_0)+\mu(\Omega')\big)^s(\mathbf{1}\otimes\mathbf{I})$ belongs to $\mathcal{B}^0\otimes\mathcal{U}(\mathfrak{X})$.

Proof. We prove by induction on s. Let P_l be a map defined by

$$P_l(\sum c_I\omega_{i_1}\circ\cdots\circ\omega_{i_s})=\sum c_I\omega_{i_1}\otimes\cdots\otimes\omega_{i_l}\wedge\omega_{i_{l+1}}\otimes\omega_{i_s}$$

and $\Omega^{(s)}=\big(\mathrm{ad}(\Omega_0)+\mu(\Omega')\big)^s(\mathbf{1}\otimes\mathbf{I})$.

For $s=0,1$, $\Omega^{(s)}\in\mathcal{B}^0\otimes\mathcal{U}(\mathfrak{X})$ is clear. For $s>1$, it is sufficient to prove $P_1(\Omega^{(s)})=0$. Since

$$\Omega^{(s)}=[\Omega_0[\Omega_0,\Omega^{(s-2)}]]+[\Omega_0,\Omega'\Omega^{(s-2)}]+\Omega'[\Omega_0,\Omega^{(s-2)}]+\Omega'\Omega'\Omega^{(s-2)}$$

and

$$P_1([\Omega_0[\Omega_0,\Omega^{(s-2)}]])=P_1([\Omega_0,\Omega'\Omega^{(s-2)}]+\Omega'[\Omega_0,\Omega^{(s-2)}])=P_1(\Omega'\Omega'\Omega^{(s-2)})$$
$$=0$$

by direct computation, we get $P_1(\Omega^{(s)})=0$. □

The fundamental solution $\mathcal{L}(z_1,z_2)$ is a grouplike element of $\widetilde{\mathcal{U}}(\mathfrak{X})$.

6.2 The decomposition theorem of the normalized fundamental solution

We consider the following four formal (generalized) 1KZ equation. Here we denote by d_{z_1} (resp. d_{z_2}) the exterior differentiation by the variable z_1 (resp. z_2).

$$d_{z_1}G(z_1,z_2) = \Omega^{(1)}_{1\otimes 2}G(z_1,z_2), \qquad \Omega^{(1)}_{1\otimes 2} = \zeta_1 Z_1 + \zeta_{11}Z_{11} + \zeta^{(1)}_{12}Z_{12}, \tag{74}$$

$$d_{z_2}G(z_2) = \Omega^{(2)}_{1\otimes 2}G(z_2), \qquad \Omega^{(2)}_{1\otimes 2} = \zeta_2 Z_2 + \zeta_{22}Z_{22}, \tag{75}$$

$$d_{z_2}G(z_1,z_2) = \Omega^{(2)}_{2\otimes 1}G(z_1,z_2), \qquad \Omega^{(2)}_{2\otimes 1} = \zeta_2 Z_2 + \zeta_{22}Z_{22} + \zeta^{(2)}_{12}Z_{12}, \tag{76}$$

$$d_{z_1}G(z_1) = \Omega^{(1)}_{2\otimes 1}G(z_1), \qquad \Omega^{(1)}_{2\otimes 1} = \zeta_1 Z_1 + \zeta_{11}Z_{11}. \tag{77}$$

The fundamental solution normalized at the origin (See Section 3.1 and 3.2) to each equation is denoted by

$$\mathcal{L}^{(i_k)}_{i_1\otimes i_2} = \hat{\mathcal{L}}^{(i_k)}_{i_1\otimes i_2}\ z_{i_k}^{Z_{i_k}},$$

$$\hat{\mathcal{L}}^{(i_k)}_{i_1\otimes i_2} = \sum_{s=0}^{\infty}\hat{\mathcal{L}}^{(i_k)}_{i_1\otimes i_2,s}, \qquad \hat{\mathcal{L}}^{(i_k)}_{i_1\otimes i_2,s}\Big|_{z_{i_k}=0} = \begin{cases} 0 & (s>0) \\ \mathbf{I} & (s=0). \end{cases}$$

Proposition 16 (The decomposition theorem for the normalized fundamental solution)**.**

(i) The fundamental solution $\mathcal{L}(z_1,z_2)$ of the formal 2KZ equation (43) normalized at the origin is decomposed to product of normalized fundamental solutions of the formal 1KZ equations as

$$\mathcal{L}(z_1,z_2) = \hat{\mathcal{L}}(z_1,z_2)z_1^{Z_1}z_2^{Z_2} = \hat{\mathcal{L}}^{(1)}_{1\otimes 2}\hat{\mathcal{L}}^{(2)}_{1\otimes 2}z_1^{Z_1}z_2^{Z_2} = \mathcal{L}^{(1)}_{1\otimes 2}\mathcal{L}^{(2)}_{1\otimes 2} \tag{78}$$

$$= \hat{\mathcal{L}}^{(2)}_{2\otimes 1}\hat{\mathcal{L}}^{(1)}_{2\otimes 1}z_1^{Z_1}z_2^{Z_2} = \mathcal{L}^{(2)}_{2\otimes 1}\mathcal{L}^{(1)}_{2\otimes 1}. \tag{79}$$

(ii) Conversely if the decomposition

$$\mathcal{L}(z_1,z_2) = G^{(i_1)}_{i_1\otimes i_2}G^{(i_2)}_{i_1\otimes i_2}$$

holds, where $G^{(i_k)}_{i_1\otimes i_2} = \hat{G}^{(i_k)}_{i_1\otimes i_2}\ z_{i_k}^{Z_{i_k}}$ is a $\mathcal{U}(\mathfrak{X}^{(i_k)}_{i_1\otimes i_2})$-valued analytic function normalized at $z_{i_k}=0$, we have $G^{(i_k)}_{i_1\otimes i_2} = \mathcal{L}^{(i_k)}_{i_1\otimes i_2}$.

Proof. (i) It is easy to prove $\mathcal{L}^{(1)}_{1\otimes 2}\mathcal{L}^{(2)}_{1\otimes 2}$ and $\mathcal{L}^{(2)}_{2\otimes 1}\mathcal{L}^{(1)}_{2\otimes 1}$ are both the normalized fundamental solution of the formal 2KZ equation.

(ii) Since $G^{(2)}_{1\otimes 2} = \hat{\mathcal{L}}(0,z_2)z_2^{Z_2}$ and $\mathcal{L}^{(2)}_{1\otimes 2}$ are both the fundamental solution of (75), $G^{(2)}_{1\otimes 2} = \mathcal{L}^{(2)}_{1\otimes 2}$ by virtue of the uniqueness of the fundamental solution. Furthermore, the uniqueness of the fundamental solution of (74) implies $G^{(1)}_{1\otimes 2}(=\mathcal{L}(z_1,z_2){G^{(2)}_{1\otimes 2}}^{-1}) = \mathcal{L}^{(1)}_{1\otimes 2}$.

□

7 The contour $C_{1\otimes 2}, C_{2\otimes 1}$ and the decomposition theorem

In this section, we interpret the decomposition theorem (Proposition 16) through the iterated integrals along the path $C_{1\otimes 2}, C_{2\otimes 1}$.

7.1 The hyperlogarithms of type $\mathcal{M}_{0,5}$ and multiple polylogarithms of two variables

We discuss hyperlogarithms which appear in the decomposition theorem.

For any words $\varphi = \zeta_1^{k_1-1} \circ \omega_1 \circ \cdots \circ \zeta_1^{k_r-1} \circ \omega_r \in S^0(A_{1\otimes 2}^{(1)})$ ($\omega_i \in \{\zeta_{11}, \zeta_{12}^{(1)}\}$), we set

$$L(\varphi; z_1) := \int_0^{z_1} \varphi = \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{\alpha_1^{n_1-n_2} \alpha_2^{n_2-n_3} \cdots \alpha_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}} z_1^{n_1} \tag{80}$$

where, in the right hand side, if $\omega_i = \zeta_{11}$ (resp. $\omega_i = \zeta_{12}^{(1)}$), $\alpha_i = 1$ (resp. $\alpha_i = z_2$). This is referred to as a hyperlogarithm of type $\mathcal{M}_{0,5}$ $L({}^{k_1}\alpha_1 \cdots {}^{k_r}\alpha_r; z_1)$.

Especially for $\omega_1 = \ldots = \omega_i = \zeta_{11}$ and $\omega_{i+1} = \ldots = \omega_{i+j} = \zeta_{12}^{(1)}$ ($\alpha_1 = \ldots = \alpha_i = 1$ and $\alpha_{i+1} = \ldots = \alpha_{i+j} = z_2$), we put

$$\begin{aligned} \mathrm{Li}_{k_1,\ldots,k_{i+j}}(i,j;z_1,z_2) &:= L({}^{k_1}1 \cdots {}^{k_i}1 {}^{k_{i+1}}z_2 \cdots {}^{k_{i+j}}z_2; z_1) \\ &= \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{z_1^{n_1} z_2^{n_{i+1}}}{n_1^{k_1} \cdots n_r^{k_r}}. \end{aligned} \tag{81}$$

We call this multiple polylogarithm of two variables of the main variable z_1 (2MPL of z_1, for short). The subscript $(k_1, \ldots, k_{i+j})$ is referred to as the index of the 2MPL and the pair of numbers (i,j) as the numbering of the 2MPL. For $j = 0$, $\mathrm{Li}_{k_1,\ldots,k_i}(i,0;z_1,z_2) = \mathrm{Li}_{k_1,\ldots,k_i}(z_1)$ is a multiple polylogarithm of one variable (see Section 3.1) and for $i = 0$, we have $\mathrm{Li}_{k_1,\ldots,k_j}(0,j;z_1,z_2) = \mathrm{Li}_{k_1,\ldots,k_j}(z_1 z_2)$.

Using the series expansion (81), we have

Proposition 17. $\mathrm{Li}_{k_1,\ldots,k_{i+j}}(i,j;z_1,z_2)$ satisfies the following differential recursive relations.

$$\begin{aligned} &\frac{\partial}{\partial z_1} \mathrm{Li}_{k_1,\ldots,k_{i+j}}(i,j;z_1,z_2) \\ &\quad = \begin{cases} \dfrac{z_2}{1 - z_1 z_2} \mathrm{Li}_{k_2,\ldots,k_j}(0, j-1; z_1, z_2) & (i = 0, k_1 = 1), \\ \dfrac{1}{1 - z_1} \mathrm{Li}_{k_2,\ldots,k_{i+j}}(i-1, j; z_1, z_2) & (i > 0, k_1 = 1), \\ \dfrac{1}{z_1} \mathrm{Li}_{k_1-1,k_2,\ldots,k_{i+j}}(i,j;z_1,z_2) & (k_1 > 1), \end{cases} \end{aligned} \tag{82}$$

$$\frac{\partial}{\partial z_2} \mathrm{Li}_{k_1,\ldots,k_{i+j}}(i,j;z_1,z_2)$$

$$
= \begin{cases} \dfrac{z_1}{1-z_1z_2}\,\mathrm{Li}_{k_2,\dots,k_j}(0,j-1;z_1,z_2) & (i=0,k_1=1), \\ \dfrac{1}{1-z_2}\,\mathrm{Li}_{k_1,\dots,k_i,k_{i+2},k_{i+j}}(i,j-1;z_1,z_2) & \\ \qquad -\dfrac{1}{1-z_2}\,\mathrm{Li}_{k_1,\dots,k_i,k_{i+2},k_{i+j}}(i-1,j;z_1,z_2) & \\ \qquad\qquad -\dfrac{1}{z_2}\,\mathrm{Li}_{k_1,\dots,k_i,k_{i+2},k_{i+j}}(i-1,j;z_1,z_2) & (i>0,k_{i+1}=1), \\ \dfrac{1}{z_2}\,\mathrm{Li}_{k_1,\dots,k_i,k_{i+1}-1,k_{i+2},\dots,k_{i+j}}(i,j;z_1,z_2) & (k_{i+1}>1). \end{cases} \tag{83}
$$

From the iterated integral expression (80), we have the following claim as for the analytic continuation of hyperlogarithms of type $\mathcal{M}_{0,5}$.

Proposition 18. Hyperlogarithms of type $\mathcal{M}_{0,5}$ is a single-valued holomorphic function in some neighborhood of $\{(z_1,z_2) \in \mathbf{P}^1 \times \mathbf{P}^1 |\ |z_1| < 1, |z_2| < 1\}$ and $(z_1,z_2) = (0,1)$. In particular, if $|z_1|$ is small enough, they can be continued analytically with respect to the parameter z_2 from $z_2 = 0$ to $z_2 = 1$.

7.2 Calculation of the iterated integral on the normalized fundamental solution and the decomposition theorem

We calculate the normalized fundamental solution of the formal 2KZ equation by iterated integral (73) on the contour $C_{1\otimes 2}$ (resp. $C_{2\otimes 1}$) and show that the result is equal to the decomposition (78) (resp. (79)).

Proposition 19.

$$
\begin{aligned}
&\int_{C_{1\otimes 2}} \big(\mathrm{ad}(\Omega_0) + \mu(\Omega')\big)^s (\mathbf{1} \otimes \mathbf{I}) \\
&\quad = \sum_{s'+s''=s} \sum_{\substack{W' \in \mathcal{W}^0_{s'}(\mathfrak{X}^{(1)}_{1\otimes 2}) \\ W'' \in \mathcal{W}^0_{s''}(\mathfrak{X}^{(2)}_{1\otimes 2})}} L(\theta^{(1)}_{1\otimes 2}(W');z_1) L(\theta^{(2)}_{1\otimes 2}(W'');z_2)\ \ \alpha(W')\alpha(W'')(\mathbf{I}),
\end{aligned} \tag{84}
$$

$$
\begin{aligned}
&\int_{C_{2\otimes 1}} \big(\mathrm{ad}(\Omega_0) + \mu(\Omega')\big)^s (\mathbf{1} \otimes \mathbf{I}) \\
&\quad = \sum_{s'+s''=s} \sum_{\substack{W' \in \mathcal{W}^0_{s'}(\mathfrak{X}^{(2)}_{2\otimes 1}) \\ W'' \in \mathcal{W}^0_{s''}(\mathfrak{X}^{(1)}_{2\otimes 1})}} L(\theta^{(2)}_{2\otimes 1}(W');z_2) L(\theta^{(1)}_{2\otimes 1}(W'');z_1)\ \ \alpha(W')\alpha(W'')(\mathbf{I}).
\end{aligned} \tag{85}
$$

Here α stands for the algebra homomorphism $\alpha : \mathcal{U}(\mathfrak{X}) \to \mathrm{End}(\mathcal{U}(\mathfrak{X}))$ defined by

$$
\alpha : (Z_1, Z_{11}, Z_2, Z_{22}, Z_{12}) \mapsto (\mathrm{ad}(Z_1), \mu(Z_{11}), \mathrm{ad}(Z_2), \mu(Z_{22}), \mu(Z_{12}))
$$

and $\theta^{(i_k)}_{i_1\otimes i_2} : \mathcal{U}(\mathfrak{X}^{(i_k)}_{i_1\otimes i_2}) \to S(A^{(i_k)}_{i_1\otimes i_2})$ is a map which gives the duality; it is defined by the displacement of letters

$$
\theta^{(i_k)}_{i_1\otimes i_2}(Z_{i_k}) = \zeta_{i_k},\ \ \theta^{(i_k)}_{i_1\otimes i_2}(Z_{i_k i_k}) = \zeta_{i_k i_k},\ \ \theta^{(i_1)}_{i_1\otimes i_2}(Z_{12}) = \zeta^{(i_1)}_{12}.
$$

Proof. We can prove this by induction on s and direct calculation. $\square$

We have the following corollary.

Corollary 20. The expression (84) (resp. (85)) is equal to the decomposition (78) (resp. (79)).

Proof. From $[Z_1, \mathcal{U}(\mathfrak{X}^{(2)}_{1\otimes 2})] = 0$, we have

$$\alpha(W')\alpha(W'')(\mathbf{I}) = \big(\alpha(W')(\mathbf{I})\big)\big(\alpha(W'')(\mathbf{I})\big)$$

for $W' \in \mathcal{W}^0(\mathfrak{X}^{(1)}_{1\otimes 2})$ and $W'' \in \mathcal{W}^0(\mathfrak{X}^{(2)}_{1\otimes 2})$. Therefore

$$\begin{aligned}&\hat{\mathcal{L}}(z_1, z_2)\\ &= \left(\sum_{W'} L(\theta^{(1)}_{1\otimes 2}(W'); z_1)\alpha(W')(\mathbf{I})\right)\left(\sum_{W''} L(\theta^{(2)}_{1\otimes 2}(W''); z_2)\alpha(W'')(\mathbf{I})\right).\end{aligned}$$

From Proposition 16 (ii), it follows that, in the right hand side above, the left factor (resp. the right factor) coincides with $\hat{\mathcal{L}}^{(1)}_{1\otimes 2}$ (resp. $\hat{\mathcal{L}}^{(2)}_{1\otimes 2}$). $\square$

7.3 The generalized harmonic product relations of hyperlogarithms

Lemma 21. Both

$$\{\alpha(W')\alpha(W'')(\mathbf{I}) \mid W' \in \mathcal{W}^0(\mathfrak{X}^{(1)}_{1\otimes 2}),\ W'' \in \mathcal{W}^0(\mathfrak{X}^{(2)}_{1\otimes 2})\}, \tag{86}$$

$$\{\alpha(W')\alpha(W'')(\mathbf{I}) \mid W' \in \mathcal{W}^0(\mathfrak{X}^{(2)}_{2\otimes 1}),\ W'' \in \mathcal{W}^0(\mathfrak{X}^{(1)}_{2\otimes 1})\} \tag{87}$$

are linearly independent sets in $\mathcal{U}(\mathfrak{X})$.

Proof. We consider the first case. Let $W' \in \mathcal{W}^0(\mathfrak{X}^{(1)}_{1\otimes 2})$ and $W'' \in \mathcal{W}^0(\mathfrak{X}^{(2)}_{1\otimes 2})$. From $[Z_1, \mathcal{W}(\mathfrak{X}^{(2)}_{1\otimes 2})] = 0$, there exists $A^i_{W'W''} \in \mathcal{U}(\mathfrak{X}^{(1)}_{1\otimes 2})$, $B^j_{W'W''} \in \mathcal{U}(\mathfrak{X}^{(2)}_{1\otimes 2})$ such that

$$\alpha(W')\alpha(W'')(\mathbf{I}) = W'W'' + \sum_{\substack{i,j\geq 0\\ (i,j)\neq(0,0)}} A^i_{W'W''} Z_1^i B^j_{W'W''} Z_2^j.$$

Since $\mathcal{U}(\mathfrak{X})$ is isomorphic to $\mathcal{U}(\mathfrak{X}^{(1)}_{1\otimes 2}) \otimes \mathcal{U}(\mathfrak{X}^{(2)}_{1\otimes 2})$ as a vector space, the set $\{W'W'' | W' \in \mathcal{W}(\mathfrak{X}^{(1)}_{1\otimes 2}),\ W'' \in \mathcal{W}(\mathfrak{X}^{(2)}_{1\otimes 2})\}$ is a linearly independent set, so is (86). $\square$

According to Proposition 19, the degree s part of $\hat{\mathcal{L}}(z_1, z_2)$ is represented as

$$\hat{\mathcal{L}}_s(z_1, z_2) = \sum_{s'+s''=s} \sum_{\substack{W' \in \mathcal{W}^0_{s'}(\mathfrak{X}^{(1)}_{1\otimes 2})\\ W'' \in \mathcal{W}^0_{s''}(\mathfrak{X}^{(2)}_{1\otimes 2})}} L(\theta^{(1)}_{1\otimes 2}(W'); z_1) L(\theta^{(2)}_{1\otimes 2}(W''); z_2)\, \alpha(W')\alpha(W'')(\mathbf{I}) \tag{88}$$

$$= \sum_{s'+s''=s} \sum_{\substack{W' \in \mathcal{W}^0_{s'}(\mathfrak{X}^{(2)}_{2\otimes 1})\\ W'' \in \mathcal{W}^0_{s''}(\mathfrak{X}^{(1)}_{2\otimes 1})}} L(\theta^{(2)}_{2\otimes 1}(W'); z_2) L(\theta^{(1)}_{2\otimes 1}(W''); z_1)\, \alpha(W')\alpha(W'')(\mathbf{I}). \tag{89}$$

As each tensor components of $\mathcal{U}(\mathfrak{X})$ of (88) and (89) are linearly independent, by comparing coefficients of $\alpha(W')\alpha(W'')(\mathbf{I})$ of both sides, we can get various relations of hyperlogarithms of type $\mathcal{M}_{0,5}$.

Putting

$$\varphi(W',W'')=\iota_{1\otimes 2}^{-1}(\theta_{1\otimes 2}^{(1)}(W')\otimes\theta_{1\otimes 2}^{(2)}(W''))\in\mathcal{B}^0$$

for $W'\in\mathcal{W}^0(\mathfrak{X}_{1\otimes 2}^{(1)}),\ W''\in\mathcal{W}^0(\mathfrak{X}_{1\otimes 2}^{(2)})$, we have

$$\int_{C_{1\otimes 2}}\iota_{1\otimes 2}(\varphi(W',W''))=L(\theta_{1\otimes 2}^{(1)}(W');z_1)L(\theta_{1\otimes 2}^{(2)}(W'');z_2), \tag{90}$$

and

$$\hat{\mathcal{L}}_s(z_1,z_2)=\sum_{s'+s''=s}\ \sum_{\substack{W'\in\mathcal{W}^0_{s'}(\mathfrak{X}_{1\otimes 2}^{(1)})\\ W''\in\mathcal{W}^0_{s''}(\mathfrak{X}_{1\otimes 2}^{(2)})}}\int_{(0,0)}^{(z_1,z_2)}\varphi(W',W'')\ \ \alpha(W')\alpha(W'')(\mathbf{I}). \tag{91}$$

Furthermore from Proposition 12, we obtain

$$\big(\operatorname{ad}(\Omega_0)+\mu(\Omega')\big)^s(\mathbf{1}\otimes\mathbf{I})=\sum_{s'+s''=s}\ \sum_{\substack{W'\in\mathcal{W}^0_{s'}(\mathfrak{X}_{1\otimes 2}^{(1)})\\ W''\in\mathcal{W}^0_{s''}(\mathfrak{X}_{1\otimes 2}^{(2)})}}\varphi(W',W'')\ \alpha(W')\alpha(W'')(\mathbf{I}). \tag{92}$$

Theorem 22 (Genelarized harmonic product relations of hyperlogarithms of type $\mathcal{M}_{0,5}$)**.** The comparison of the coefficients in the decomposition (78) and (79) is equal to

$$L(\theta_{1\otimes 2}^{(1)}(W');z_1)L(\theta_{1\otimes 2}^{(2)}(W'');z_2)=\int_{C_{2\otimes 1}}\iota_{2\otimes 1}(\varphi(W',W'')) \tag{93}$$

for each $W'\in\mathcal{W}^0(\mathfrak{X}_{1\otimes 2}^{(1)}),\ W''\in\mathcal{W}^0(\mathfrak{X}_{1\otimes 2}^{(2)})$.

We call (93) the generalized harmonic product relations of hyperlogarithms of type $\mathcal{M}_{0,5}$. In Section 8.2, we show the system of the relations contains properly the harmonic product of 2MPLs.

7.4 Examples of the generalized harmonic product relations

We show some examples of the generalized harmonic product relations.

7.4.1 The relations generated by $\hat{\mathcal{L}}_2(z_1,z_2)$

Computing the iterated integral of $\hat{\mathcal{L}}_2(z_1,z_2)$ along the contour $C_{1\otimes 2}$, we obtain

$$\begin{aligned}\hat{\mathcal{L}}_2(z_1,z_2)\overset{C_{1\otimes 2}}{=}&\operatorname{Li}_2(z_1)[Z_1,Z_{11}]+\operatorname{Li}_2(z_1z_2)[Z_1,Z_{12}]+\operatorname{Li}_{1,1}(z_1)Z_{11}^2\\&+\operatorname{Li}_{1,1}(1,1;z_1,z_2)Z_{11}Z_{12}+L({}^1{z_2}^1 1;z_1)Z_{12}Z_{11}+\operatorname{Li}_{1,1}(z_1z_2)Z_{12}^2\\&+\operatorname{Li}_1(z_1)\operatorname{Li}_1(z_2)Z_{11}Z_{22}+\operatorname{Li}_1(z_1z_2)\operatorname{Li}_1(z_2)Z_{12}Z_{22}\\&+\operatorname{Li}_2(z_2)[Z_2,Z_{22}]+\operatorname{Li}_{1,1}(z_2)Z_{22}^2.\end{aligned}$$

This certainly coincides with the degree 2 part of the decomposition $\hat{\mathcal{L}}_{1\otimes 2}^{(1)}\hat{\mathcal{L}}_{1\otimes 2}^{(2)}$, and yields the following two nontrivial relations of hyperlogarithms.

- The coefficients of $Z_{11}Z_{12}$

  $\varphi(Z_{11}Z_{12},\mathbf{I})=\zeta_{11}\circ\zeta_{12}+\zeta_{22}\circ\zeta_{11}-\zeta_{22}\circ\zeta_{12}-\zeta_2\circ\zeta_{12}$ and the relation is

$$\mathrm{Li}_{1,1}(1,1;z_1,z_2)=\mathrm{Li}_1(z_2)\,\mathrm{Li}_1(z_1)-\mathrm{Li}_{1,1}(1,1;z_2,z_1)-\mathrm{Li}_2(0,1;z_2,z_1). \tag{94}$$

- The coefficients of $Z_{12}Z_{11}$

  $\varphi(Z_{12}Z_{11},\mathbf{I})=\zeta_{12}\circ\zeta_{11}-\zeta_{22}\circ\zeta_{11}+\zeta_{22}\circ\zeta_{12}+\zeta_2\circ\zeta_{12}$ and the relation is

$$\begin{aligned}L({}^1z_2{}^11;z_1)&=\mathrm{Li}_1(0,1;z_2,z_1)\,\mathrm{Li}_1(z_1)-\mathrm{Li}_1(z_2)\,\mathrm{Li}_1(z_1)\\&\quad+\mathrm{Li}_{1,1}(1,1;z_2,z_1)+\mathrm{Li}_2(0,1;z_2,z_1).\end{aligned} \tag{95}$$

  This relation is obtained by the shuffle product $\zeta_{12}⧢\zeta_{11}=\zeta_{12}\circ\zeta_{11}+\zeta_{11}\circ\zeta_{12}$ and (94).

We note that the coefficients of $Z_{12}Z_{22}$ also yields nontrivial relation, but, the relation can be obtained by interchanging the variable z_1 and z_2 in the previous two relations.

7.4.2 The relation of the coefficients of $\alpha(Z_{12}Z_{11}Z_{12})\alpha(\mathbf{I})(\mathbf{I})$

The coefficient of $\alpha(Z_{12}Z_{11}Z_{12})\alpha(\mathbf{I})(\mathbf{I})$ in (84) is $L({}^1z_2{}^11^1z_2;z_1)$ and we obtain

$$\begin{aligned}\varphi(Z_{12}Z_{11}Z_{12},\mathbf{I})&=\zeta_{12}\circ\zeta_{11}\circ\zeta_{12}+\zeta_{12}\circ\zeta_{22}\circ\zeta_{11}-\zeta_{12}\circ\zeta_{22}\circ\zeta_{12}\\&\quad-\zeta_{12}\circ\zeta_2\circ\zeta_{12}-\zeta_{22}\circ\zeta_{11}\circ\zeta_{12}+\zeta_{22}\circ\zeta_{12}\circ\zeta_{11}\\&\quad+2\zeta_{22}\circ\zeta_2\circ\zeta_{12}-2\zeta_{22}\circ\zeta_{22}\circ\zeta_{11}+2\zeta_{22}\circ\zeta_{22}\circ\zeta_{12}.\end{aligned}$$

The generalized harmonic product relation of this case is

$$\begin{aligned}L({}^1z_2{}^11^1z_2;z_1)&=-2\,\mathrm{Li}_{1,1}(z_2)\,\mathrm{Li}_1(z_1)+2\,\mathrm{Li}_{1,1,1}(2,1;z_2,z_1)+2\,\mathrm{Li}_{1,2}(1,1;z_2,z_1)\\&\quad+L({}^1z_1{}^11;z_2)\,\mathrm{Li}_1(z_1)+\mathrm{Li}_{1,1}(1,1;z_2,z_1)\,\mathrm{Li}_1(z_1)\\&\quad-L({}^1z_1{}^11^1z_1;z_2)-\mathrm{Li}_{1,2}(0,2;z_2,z_1).\end{aligned}$$

7.5 Connection problem for the solutions of the formal 2KZ equation and the decomposition theorem

We denote by $\mathcal{L}^{(0,0)}(z_1,z_2)=\mathcal{L}(z_1,z_2)$ the fundamental solution of the formal 2KZ equation normalized at $(z_1,z_2)=(0,0)$ and

$$\mathcal{L}^{(1,0)}(z_1,z_2)=\hat{\mathcal{L}}^{(1,0)}(z_1,z_2)(1-z_1)^{-Z_{11}}z_2^{Z_2}$$

normalized at $(1,0)$, where $\hat{\mathcal{L}}^{(1,0)}(z_1,z_2)$ is holomorphic on some neighborhood of $(1,0)$ and $\hat{\mathcal{L}}^{(1,0)}(1,0)=\mathbf{I}$. According to Proposition 16, we can write $\mathcal{L}^{(0,0)}(z_1,z_2)$ and $\mathcal{L}^{(1,0)}(z_1,z_2)$ as

$$\begin{aligned}\mathcal{L}^{(0,0)}(z_1,z_2)&=\mathcal{L}^{(2),(0,0)}_{2\otimes1}\mathcal{L}^{(1),(0,0)}_{2\otimes1},\\\mathcal{L}^{(1,0)}(z_1,z_2)&=\mathcal{L}^{(2),(1,0)}_{2\otimes1}\mathcal{L}^{(1),(1,0)}_{2\otimes1},\end{aligned}$$

where, $\mathcal{L}^{(2),(0,0)}_{2\otimes1}$ (resp. $\mathcal{L}^{(2),(1,0)}_{2\otimes1}$) is $\mathcal{U}(\mathfrak{X}^{(2)}_{2\otimes1})$-valued analytic function normalized at $z_2=0$ and $\mathcal{L}^{(1),(0,0)}_{2\otimes1}$ (resp. $\mathcal{L}^{(1),(1,0)}_{2\otimes1}$) is $\mathcal{U}(\mathfrak{X}^{(1)}_{2\otimes1})$-valued analytic function normalized at $z_1=0$ (resp. $z_1=1$).

By virtue of Proposition 16 (ii), Proposition 18 and the connection problem for the formal 1KZ equation, we have

$$\mathcal{L}_{2\otimes 1}^{(2),(0,0)} = \mathcal{L}_{2\otimes 1}^{(2),(1,0)},$$
$$\mathcal{L}_{2\otimes 1}^{(1),(0,0)} = \mathcal{L}_{2\otimes 1}^{(1),(1,0)}\Phi_{\mathrm{KZ}}(Z_1, Z_{11}),$$

where $\Phi_{\mathrm{KZ}}(Z_1, Z_{11})$ stands for the Drinfel'd associator of the variables Z_1 and Z_{11} [D]. Therefore we obtain

Proposition 23. The connection formula between the fundamental solutions of the formal 2KZ equation (43) normalized at $(z_1, z_2) = (0,0)$ and $(1,0), (0,1)$ is given as

$$\mathcal{L}^{(0,0)}(z_1, z_2) = \mathcal{L}^{(1,0)}(z_1, z_2)\Phi_{\mathrm{KZ}}(Z_1, Z_{11}), \tag{96}$$
$$\mathcal{L}^{(0,0)}(z_1, z_2) = \mathcal{L}^{(0,1)}(z_1, z_2)\Phi_{\mathrm{KZ}}(Z_2, Z_{22}). \tag{97}$$

As considered above, the connection problem of the formal 2KZ equation reduces to the decomposition theorem (Proposition 16) and the connection problem of the formal 1KZ equation.

The connection problem for fundamental solutions of the formal 2KZ equation and its application will be considered in the forthcoming paper [OU1].

8 The harmonic product of multiple zeta values and the decomposition theorem

In this section, we discuss the harmonic product of multiple zeta values and multiple polylogarithms, and show that the generalized harmonic product relations contain the harmonic product of multiple polylogarithms.

8.1 Harmonic product of the multiple zeta values and multiple polylogarithms

Let $\mathcal{I}$ be a $\mathbf{C}$-vector space spanned by all indexes (sequences of $\mathbf{Z}_{>0}$) and $\cdot$ be a product of $\mathcal{I}$ by concatenation as $(k_1, \ldots, k_i)\cdot(l_1, \ldots, l_j) = (k_1, \ldots, k_i, l_1, \ldots, l_j)$. We introduce the harmonic product $*$ of $\mathcal{I}$ defined by

$$\begin{aligned}
&\mathbf{k} * \emptyset = \emptyset * \mathbf{k} = \mathbf{k} \quad (\mathbf{k} \in \mathcal{I}),\\
&(k_1, \ldots, k_i) * (l_1, \ldots, l_j) = (k_1) \cdot ((k_2, \ldots, k_i) * (l_1, \ldots, l_j))\\
&\qquad\qquad + (l_1) \cdot ((k_1, \ldots, k_i) * (l_2, \ldots, l_j))\\
&\qquad\qquad\qquad + (k_1 + l_1) \cdot ((k_2, \ldots, k_i) * (l_2, \ldots, l_j)),
\end{aligned}$$

where $\emptyset$ stands for the empty index. Then $(\mathcal{I}, *)$ becomes a commutative algebra. This is nothing but the harmonic algebra structure due to Hoffman [Ho].

Lemma 24.

$$\begin{aligned}
&(k_1,\dots,k_i) * (l_1,\dots,l_j)\\
&\qquad = \sum_{p=0}^{j-1} \Big((l_1,\dots,l_p,k_1)\cdot((k_2,\dots,k_i)*(l_{p+1},\dots,l_j))\\
&\qquad\qquad + (l_1,\dots,l_p,k_1+l_{p+1})\cdot((k_2,\dots,k_i)*(l_{p+2},\dots,l_j))\Big)\\
&\qquad\quad + (l_1,\dots,l_j,k_1,\dots,k_i),
\end{aligned} \tag{98}$$

where we regard an incorrect index as the empty index $\emptyset$. For instance, $(l_{j+1},\dots,l_j)=\emptyset$ and $(l_1,\dots,l_0,k_1)=(k_1)$.

Proof. The lemma can be prove easily by induction on j. $\square$

The harmonic product of the multiple zeta values (36) is the relation

$$\zeta(k_1,\dots,k_i)\zeta(l_1,\dots,l_j)=\zeta((k_1,\dots,k_i)*(l_1,\dots,l_j)),$$

for $k_1,l_1>1$ (the multiple zeta values ζ is extended to $\mathcal{I}$ linearly), and likewise one can define the harmonic product of multiple polylogarithms by

$$\begin{aligned}
&\mathrm{Li}_{k_1,\dots,k_i}(z_1)\,\mathrm{Li}_{l_1,\dots,l_j}(z_2)\\
&\qquad = \sum_{p=1}^{i-1}\Big(\mathrm{Li}_{(k_1,\dots,k_p,l_1)\cdot((k_{p+1},\dots,k_i)*(l_2,\dots,l_j))}(p,\bullet;z_1,z_2)\\
&\qquad\qquad + \mathrm{Li}_{(k_1,\dots,k_p,l_1+k_{p+1})\cdot((k_{p+2},\dots,k_i)*(l_2,\dots,l_j))}(p,\bullet;z_1,z_2)\Big)\\
&\qquad\quad + \mathrm{Li}_{(k_1,\dots,k_i,l_1,\dots,l_j)}(i,j;z_1,z_2)\\
&\qquad + \sum_{p=1}^{j-1}\Big(\mathrm{Li}_{(l_1,\dots,l_p,k_1)\cdot((k_2,\dots,k_i)*(l_{p+1},\dots,l_j))}(p,\bullet;z_2,z_1)\\
&\qquad\qquad + \mathrm{Li}_{(l_1,\dots,l_p,k_1+l_{p+1})\cdot((k_2,\dots,k_i)*(l_{p+2},\dots,l_j))}(p,\bullet;z_2,z_1)\Big)\\
&\qquad\quad + \mathrm{Li}_{(l_1,\dots,l_j,k_1,\dots,k_i)}(j,i;z_2,z_1)\\
&\qquad + \mathrm{Li}_{(k_1+l_1)\cdot((k_2,\dots,k_i)*(l_2,\dots,l_j))}(0,\bullet;z_2,z_1),
\end{aligned} \tag{99}$$

where $\mathrm{Li}_{\mathbf{k}}$ is extended to $\mathcal{I}$ linearly and the second numbering $\bullet$ means (the length of index) $-$ (the first numbering). Namely, the harmonic product of $\mathrm{Li}_{k_1,\dots,k_i}(z_1)\,\mathrm{Li}_{l_1,\dots,l_j}(z_2)$ is expressed as

$$\mathrm{Li}_{k_1,\dots,k_i}(z_1)\,\mathrm{Li}_{l_1,\dots,l_j}(z_2)=\sum_{\mathbf{k}\in\mathcal{I}((k_1,\dots,k_i)*(l_1,\dots,l_j))} c_{\mathbf{k}}\ \underline{\mathrm{Li}_{\mathbf{k}}}(z_1,z_2),$$

where $\mathcal{I}((k_1,\dots,k_i)*(l_1,\dots,l_j))$ is the set of indexes and $c_{\mathbf{k}}\in\mathbf{Z}_{>0}$ are positive integers defined as

$$(k_1,\dots,k_i)*(l_1,\dots,l_j)=\sum_{\mathbf{k}\in\mathcal{I}((k_1,\dots,k_i)*(l_1,\dots,l_j))} c_{\mathbf{k}}\ \mathbf{k}$$

by harmonic product of indexes, and the 2MPL $\underline{\mathrm{Li}_{\mathbf{k}}}(z_1,z_2)$ stands for

$$\underline{\mathrm{Li}_{\mathbf{k}}}(z_1, z_2)$$
$$= \begin{cases} \mathrm{Li}_{\mathbf{k}}(\,(\text{the position of } l_1 \text{ in } \mathbf{k}) - 1, \bullet; z_1, z_2) & (\mathbf{k} \text{ starts from } k_1), \\ \mathrm{Li}_{\mathbf{k}}(\,(\text{the position of } k_1 \text{ in } \mathbf{k}) - 1, \bullet; z_2, z_1) & (\mathbf{k} \text{ starts from } l_1), \\ \mathrm{Li}_{\mathbf{k}}(z_1 z_2) & (\mathbf{k} \text{ starts from } k_1 + l_1). \end{cases}$$

For $i = j = 1$ in (99), the harmonic product of $\mathrm{Li}_{k_1}(z_1)\,\mathrm{Li}_{l_1}(z_2)$ is written as

$$\mathrm{Li}_{k_1}(z_1)\,\mathrm{Li}_{l_1}(z_2) = \mathrm{Li}_{k_1,l_1}(1,1; z_1, z_2) + \mathrm{Li}_{k_1+l_1}(z_1 z_2) + \mathrm{Li}_{l_1,k_1}(1,1; z_2, z_1).$$

This is equal to

$$\sum_{m>0} \frac{z_1^m}{m^{k_1}} \sum_{n>0} \frac{z_2^n}{n^{l_1}} = \left(\sum_{m>n>0} + \sum_{m=n>0} + \sum_{n>m>0} \right) \frac{z_1^m z_2^n}{m^{k_1} n^{l_1}}$$

by series expression.

8.2 Relationship between the generalized harmonic product relations of hyperlogarithms and the harmonic product of MPLs

We show that the generalized harmonic product relations of hyperlogarithms of type $\mathcal{M}_{0,5}$ given by Theorem 22 involve properly all of the harmonic products of multiple polylogarithms.

We denote by $\mathcal{M}PL(z_1, z_2)$ the $\mathbf{C}$-linear space whose basis is the set of all 2MPLs of z_1, and $\mathcal{M}PL_s(z_1, z_2)$ the subspace spanned by 2MPLs whose weight (= the sum of indexes) is s.

We define the linear operator $\mathfrak{d} : \mathcal{M}PL(z_1, z_2) \otimes S(A) \to \mathcal{M}PL(z_1, z_2) \otimes S(A)$ by

$$\mathfrak{d}(\mathrm{Li}_{k_1,\ldots,k_{i+j}}(i, j; z_1, z_2) \otimes \omega) = \omega \circ (d\,\mathrm{Li}_{k_1,\ldots,k_{i+j}}(i, j; z_1, z_2)) \tag{100}$$

for $\mathrm{Li}_{k_1,\ldots,k_{i+j}}(i, j; z_1, z_2) \otimes \omega \in \mathcal{M}PL_k(z_1, z_2) \otimes S_s(A)$ $(k = k_1 + \cdots + k_{i+j})$. Proposition 17 says that the right hand side of (100) belongs to $\mathcal{M}PL_{k-1}(z_1, z_2) \otimes S_{s+1}(A)$. Moreover we can get the following lemma and corollary.

Lemma 25.

$$\mathfrak{d}(\mathcal{M}PL_k(z_1, z_2) \otimes \mathcal{B}_s) \subset \mathcal{M}PL_{k-1}(z_1, z_2) \otimes \mathcal{B}_{s+1}. \tag{101}$$

Proof. By induction on k. □

This lemma says that

$$\int_{(0,0)}^{(z_1,z_2)} \mathfrak{d}^s(\mathrm{Li}_{k_1,\ldots,k_{i+j}}(i, j; z_1, z_2) \otimes \mathbf{1}) = \mathrm{Li}_{k_1,\ldots,k_{i+j}}(i, j; z_1, z_2)$$

for all $1 \le s \le k_1 + \cdots + k_{i+j}$. Especially for $s = k_1 + \cdots + k_{i+j}$, we obtain

$$\varphi = \mathfrak{d}^{k_1+\cdots+k_{i+j}}(\mathrm{Li}_{k_1,\ldots,k_{i+j}}(i, j; z_1, z_2) \otimes \mathbf{1}) \in \mathcal{B} \tag{102}$$

and $\displaystyle\int \varphi = \mathrm{Li}_{k_1,\ldots,k_{i+j}}(i, j; z_1, z_2)$.

Corollary 26. Let $W = Z_1^{k_1-1}Z_{11}\cdots Z_1^{k_i-1}Z_{11}Z_1^{k_{i+1}-1}Z_{12}\cdots Z_1^{k_{i+j}-1}Z_{12}$ be a word of $\mathcal{U}(\mathfrak{X}^{(1)}_{1\otimes 2})$. Then we have

$$\varphi(W,\mathbf{I}) = \mathfrak{d}^{k_1+\cdots+k_{i+j}}(\mathrm{Li}_{k_1,\dots,k_{i+j}}(i,j;z_1,z_2)\otimes \mathbf{1}).$$

Moreover the generalized harmonic product relation $\displaystyle\int_{C_{1\otimes 2}}\varphi(W,\mathbf{I}) = \int_{C_{2\otimes 1}}\varphi(W,\mathbf{I})$ is expressed as a recursive formula

$$\mathrm{Li}_{k_1,\dots,k_{i+j}}(i,j;z_1,z_2) = \int_{C_{2\otimes 1}} d\,\mathrm{Li}_{k_1,\dots,k_{i+j}}(i,j;z_1,z_2). \tag{103}$$

Proof. The first claim is clear by Proposition 12. The second one immediately follows from Theorem 22 for $\varphi(W,\mathbf{I})$. □

That is to say, since the total differential $d\,\mathrm{Li}_{k_1,\dots,k_{i+j}}(i,j;z_1,z_2)$ in the right hand side of (103) belongs to $\mathcal{MPL}_{k_1+\cdots+k_{i+j}-1}\otimes\mathcal{B}_1$, we can apply (103) again to the $\mathcal{MPL}$ parts. Therefore we can compute the iterated integral $\displaystyle\int_{C_{2\otimes 1}}\varphi(W,\mathbf{I})$ along the path $C_{2\otimes 1}$ recursively.

Theorem 27. The relations (103) of Corollary 26 and the harmonic products of MPLs (99) are equivalent.

In what follows, we prove Theorem 27.

Lemma 28. The relations (106) $(i\ge 0, j>0)$ and (107) $(i>0, j\ge 0)$ are equivalent to

$$\begin{aligned}\int_0^{z_2}&\underbrace{\frac{dz_2}{z_2}\circ\cdots\circ\frac{dz_2}{z_2}}_{k-1\text{ times}}\circ\frac{dz_2}{1-z_2}\,\mathrm{Li}_{k_1,\dots,k_{i+j}}(i,j;z_1,z_2)\\
&=\sum_{s=0}^{i-1}\Big(\mathrm{Li}_{k_1,\dots,k_{i-s},k,k_{i-s+1},\dots,k_{i+j}}(i-s,j+s+1;z_1,z_2)\\
&\qquad+\mathrm{Li}_{k_1,\dots,k_{i-s}+k,k_{i-s+1},\dots,k_{i+j}}(i-s-1,j+s+1;z_1,z_2)\Big)\\
&\qquad+\mathrm{Li}_{k,k_1,\dots,k_{i+j}}(1,i+j;z_2,z_1)\end{aligned} \tag{104}$$

for $i,j\ge 0$.

Proof. From Proposition 17, we have

$$\begin{aligned}\mathrm{Li}_{k_1,\dots,k_{i+j}}(i,j;z_1,z_2) &= \int_{C_{2\otimes 1}} d\,\mathrm{Li}_{k_1,\dots,k_{i+j}}(i,j;z_1,z_2)\\
&=\int_0^{z_2}\frac{\partial}{\partial z_2}\mathrm{Li}_{k_1,\dots,k_{i+j}}(i,j;z_1,z_2)dz_2\end{aligned} \tag{105}$$

for $i,j>0$. As the relations (103) with $i=0$ or $j=0$ are trivial, this implies that the system of relations (103) and (105) are equivalent.

Next we have

$$\int_0^{z_2}\frac{dz_2}{z_2}\,\mathrm{Li}_{k_1,\dots,k_{i+j}}(i,j;z_1,z_2) = \mathrm{Li}_{k_1,\dots,k_{i+1}+1,\dots,k_{i+j}}(i,j;z_1,z_2) \tag{106}$$

for $i \geq 0, j > 0$ and

$$\int_0^{z_2} \frac{dz_2}{1-z_2} \operatorname{Li}_{k_1,\dots,k_{i+j}}(i,j;z_1,z_2)$$
$$= \sum_{s=0}^{i-1} \Big(\operatorname{Li}_{k_1,\dots,k_{i-s},1,k_{i-s+1},\dots,k_{i+j}}(i-s,j+s+1;z_1,z_2)$$
$$+ \operatorname{Li}_{k_1,\dots,k_{i-s}+1,k_{i-s+1},\dots,k_{i+j}}(i-s-1,j+s+1;z_1,z_2) \Big)$$
$$+ \operatorname{Li}_{1,k_1,\dots,k_{i+j}}(1,i+j;z_2,z_1). \quad (107)$$

for $i > 0, j \geq 0$ by direct calculation. Moreover, all of these relations (106) and (107) can be written as linear combinations of relations (105), vise versa.

Therefore it is enough to show that the set of relations (106) and (107) is equivalent to (104). It is clear that (104) for $i > 0$ follows from (106), (107). For $i = 0$, the relation (104) is trivial. Conversely, that (107) follows from (104) is clear. We can show that (106) follows from (104) by induction on i. □

Proof of Theorem 27.
Step 1. We prove the harmonic product of MPLs (99) from (104). By iterated integral expression of MPLs (33), we have

$$\operatorname{Li}_{k_1,\dots,k_i}(z_1) \operatorname{Li}_{l_1,\dots,l_j}(z_2)$$
$$= \int_0^{z_2} \underbrace{\frac{dz_2}{z_2} \circ \cdots \circ \frac{dz_2}{z_2}}_{l_1-1 \text{ times}} \circ \frac{dz_2}{1-z_2} \circ \cdots \circ \underbrace{\frac{dz_2}{z_2} \circ \cdots \circ \frac{dz_2}{z_2}}_{l_j-1 \text{ times}} \circ \frac{dz_2}{1-z_2} \operatorname{Li}_{k_1,\dots,k_i}(i,0;z_1,z_2). \tag{108}$$

Applying (104) to (108), we show by induction on j that the right hand side of (108) can be written as the right hand side of (99).

For $j = 1$, it is clear. In general case, we assume that

$$\int_0^{z_2} \underbrace{\frac{dz_2}{z_2} \circ \cdots \circ \frac{dz_2}{z_2}}_{l_2-1 \text{ times}} \circ \frac{dz_2}{1-z_2} \circ \cdots \circ \underbrace{\frac{dz_2}{z_2} \circ \cdots \circ \frac{dz_2}{z_2}}_{l_j-1 \text{ times}} \circ \frac{dz_2}{1-z_2} \operatorname{Li}_{k_1,\dots,k_i}(i,0;z_1,z_2)$$
$$= \big(\text{the right hand side of (99) for } \operatorname{Li}_{k_1,\dots,k_i}(z_1) \operatorname{Li}_{l_2,\dots,l_j}(z_2)\big).$$

From the assumption, we have

$$\int_0^{z_2} \underbrace{\frac{dz_2}{z_2} \circ \cdots \circ \frac{dz_2}{z_2}}_{l_1-1 \text{ times}} \circ \frac{dz_2}{1-z_2} \circ \cdots \circ \underbrace{\frac{dz_2}{z_2} \circ \cdots \circ \frac{dz_2}{z_2}}_{l_j-1 \text{ times}} \circ \frac{dz_2}{1-z_2} \operatorname{Li}_{k_1,\dots,k_i}(i,0;z_1,z_2)$$
$$= \int_0^{z_2} \underbrace{\frac{dz_2}{z_2} \circ \cdots \circ \frac{dz_2}{z_2}}_{l_1-1 \text{ times}} \circ \frac{dz_2}{1-z_2}$$
$$\big(\text{the right hand side of (99) for } \operatorname{Li}_{k_1,\dots,k_i}(z_1) \operatorname{Li}_{l_2,\dots,l_j}(z_2)\big)$$

$$
\begin{aligned}
&= \sum_{p=1}^{i-1} \Big(\int_0^{z_2} \underbrace{\frac{dz_2}{z_2} \circ \cdots \circ \frac{dz_2}{z_2}}_{l_1-1 \text{ times}} \circ \frac{dz_2}{1-z_2} \mathrm{Li}_{(k_1,\dots,k_p,l_2)\cdot((k_{p+1},\dots,k_i)*(l_3,\dots,l_j))}(p,\bullet;z_1,z_2) \\
&\qquad + \int_0^{z_2} \underbrace{\frac{dz_2}{z_2} \circ \cdots \circ \frac{dz_2}{z_2}}_{l_1-1 \text{ times}} \circ \frac{dz_2}{1-z_2} \\
&\qquad\qquad \mathrm{Li}_{(k_1,\dots,k_p,k_{p+1}+l_2)\cdot((k_{p+2},\dots,k_i)*(l_3,\dots,l_j))}(p,\bullet;z_1,z_2)\Big) \\
&+ \int_0^{z_2} \underbrace{\frac{dz_2}{z_2} \circ \cdots \circ \frac{dz_2}{z_2}}_{l_1-1 \text{ times}} \circ \frac{dz_2}{1-z_2} \mathrm{Li}_{k_1,\dots,k_i,l_2,\dots,l_j}(i,j-1;z_1,z_2) \\
&+ \sum_{p=2}^{j} \Big(\mathrm{Li}_{(l_1,\dots,l_p,k_1)\cdot((k_2,\dots,k_i)*(l_{p+1},\dots,l_j))}(p,\bullet;z_2,z_1) \\
&\qquad + \mathrm{Li}_{(l_1,\dots,l_p,k_1+l_{p+1})\cdot((k_2,\dots,k_i)*(l_{p+2},\dots,l_j))}(p,\bullet;z_2,z_1)\Big) \\
&+ \mathrm{Li}_{l_1,\dots,l_j,k_1,\dots,k_i}(j,i;z_2,z_1) \\
&+ \mathrm{Li}_{(l_1,k_1+l_2)\cdot((k_2,\dots,k_i)*(l_3,\dots,l_j))}(1,\bullet;z_2,z_1).
\end{aligned}
$$

Here we expand this by using (104) and denote by $f_0(z_1,z_2)+f_1(z_1,z_2)+f_2(z_1,z_2)$ the result, where $f_1(z_1,z_2)$ (resp. $f_2(z_1,z_2), f_0(z_1,z_2)$) is a sum of 2MPLs of z_1 (resp. 2MPLs of z_2, MPLs of z_1z_2). By easy calculation, we have

$$
\begin{aligned}
f_1(z_1,z_2) &= \sum_{p=1}^{i-1}\sum_{q=p}^{i-1} \Big(\mathrm{Li}_{(k_1,\dots,k_p,l_1,k_{p+1},\dots,k_q,l_2)\cdot((k_{q+1},\dots,k_i)*(l_3,\dots,l_j))}(p,\bullet;z_1,z_2) \\
&\qquad + \mathrm{Li}_{(k_1,\dots,k_p,l_1,k_{p+1},\dots,k_q,k_{q+1}+l_2)\cdot((k_{q+2},\dots,k_i)*(l_3,\dots,l_j))}(p,\bullet;z_1,z_2)\Big) \\
&+ \sum_{p=1}^{i} \mathrm{Li}_{(k_1,\dots,k_p,l_1,k_{p+1},\dots,k_i,l_2,\dots,l_j}(p,\bullet;z_1,z_2) \\
&+ \sum_{p=1}^{i-2}\sum_{q=p+1}^{i-1} \Big(\mathrm{Li}_{(k_1,\dots,k_{p+1}+l_1,k_{p+2},\dots,k_q,l_2)\cdot((k_{q+1},\dots,k_i)*(l_3,\dots,l_j))}(p,\bullet;z_1,z_2) \\
&\qquad + \mathrm{Li}_{(k_1,\dots,k_{p+1}+l_1,k_{p+2},\dots,k_q,k_{q+1}+l_2)\cdot((k_{q+2},\dots,k_i)*(l_3,\dots,l_j))}(p,\bullet;z_1,z_2)\Big) \\
&+ \sum_{p=1}^{i-1} \mathrm{Li}_{k_1,\dots,k_{p+1}+l_1,k_{p+2},\dots,k_i,l_2,\dots,l_j}(p,\bullet;z_1,z_2) \\
&= \sum_{p=1}^{i-1} \Big(\mathrm{Li}_{(k_1,\dots,k_p,l_1)\cdot((k_{p+1},\dots,k_i)*(l_2,\dots,l_j))}(p,\bullet,z_1,z_2) \\
&\qquad + \mathrm{Li}_{(k_1,\dots,k_p,l_1+k_{p+1})\cdot((k_{p+2},\dots,k_i)*(l_2,\dots,l_j))}(p,\bullet,z_1,z_2)\Big) \\
&+ \mathrm{Li}_{k_1,\dots,k_i,l_1,\dots,l_j}(i,j;z_1,z_2),
\end{aligned}
$$

where the second equation is due to Lemma 24. Likewise we have

$$\begin{aligned} f_2(z_1,z_2) = \sum_{p=1}^{j-1} \Big(&\mathrm{Li}_{(l_1,\dots,l_p,k_1)\cdot((k_2,\dots,k_i)*(l_{p+1},\dots,l_j))}(p,\bullet;z_2,z_1) \\ &+ \mathrm{Li}_{(l_1,\dots,l_p,k_1+l_{p+1})\cdot((k_2,\dots,k_i)*(l_{p+2},\dots,l_j))}(p,\bullet;z_2,z_1)\Big) \\ &+ \mathrm{Li}_{l_1,\dots,l_j,k_1,\dots,k_i}(j,i;z_2,z_1), \\ f_0(z_1,z_2) = \mathrm{Li}&_{(k_1+l_1)\cdot((k_2,\dots,k_i)*(l_2,\dots,l_j))}(0,\bullet;z_2,z_2). \end{aligned}$$

Therefore $f_0(z_1,z_2)+f_1(z_1,z_2)+f_2(z_1,z_2)$ is nothing but the right hand side of (99) for $\mathrm{Li}_{k_1,\dots,k_i}(z_1)\,\mathrm{Li}_{l_1,\dots,l_j}(z_2)$.

Step 2. Next we show the converse statement of (i). Assume (99). We define the total order

$$\begin{aligned} &(i,j) < (i',j') \qquad && (i+j < i'+j'), \\ &(i,j) < (i',j') \qquad && (i+j = i'+j',\;\; i<i') \end{aligned} \tag{109}$$

on numbering (i,j) and denote the harmonic product by

$$\begin{aligned} \mathrm{Li}_{k_1,\dots,k_i}(z_1)\,\mathrm{Li}_{l_1,\dots,l_j}(z_2) = &\mathrm{Li}_{k_1,\dots,k_i,l_1,\dots,l_j}(i,j;z_1,z_2) \\ &+ F_{(k_1,\dots,k_i),(l_1,\dots,l_j)}(z_1,z_2) \\ &+ G_{(k_1,\dots,k_i),(l_1,\dots,l_j)}(z_1,z_2), \end{aligned}$$

where F is the sum of all terms of 2MPL of z_1 whose numbering is less than (i,j) and G is the sum of all terms of 2MPL of z_2. In this notation, we can show the equation

$$\begin{aligned} &\int_0^{z_2} \underbrace{\frac{dz_2}{z_2}\circ\cdots\circ\frac{dz_2}{z_2}}_{l_1-1}\circ\frac{dz_2}{1-z_2}\,\mathrm{Li}_{k_1,\dots,k_i,l_2,\dots,l_j}(i,j-1;z_1,z_2) \\ &= \mathrm{Li}_{k_1,\dots,k_i,l_1,\dots,l_j}(i,j;z_1,z_2) \\ &+ F_{(k_1,\dots,k_i),(l_1,\dots,l_j)}(z_1,z_2) \\ &+ G_{(k_1,\dots,k_i),(l_1,\dots,l_j)}(z_1,z_2) \\ &- \int_0^{z_2} \underbrace{\frac{dz_2}{z_2}\circ\cdots\circ\frac{dz_2}{z_2}}_{l_1-1}\circ\frac{dz_2}{1-z_2} F_{(k_1,\dots,k_i),(l_2,\dots,l_j)}(z_1,z_2) \\ &- \int_0^{z_2} \underbrace{\frac{dz_2}{z_2}\circ\cdots\circ\frac{dz_2}{z_2}}_{l_1-1}\circ\frac{dz_2}{1-z_2} G_{(k_1,\dots,k_i),(l_2,\dots,l_j)}(z_1,z_2) \end{aligned}$$

restore the relations (104) by induction on this total order. We have thus completed the proof of Theorem 27. □

References

[A] V.I.Arnold, The cohomology ring of the colored braid group, Mat. Zametki **5** (1969), pp. 227-231: Math. Notes **5** (1969), pp. 138-140.

[B] F.Brown, Multiple zeta values and periods of moduli spaces $\overline{\mathfrak{M}}_{0,n}(\mathbf{R})$, preprint (2006) arXiv:math/0606419v1.

[BF] A.Besser and H.Furusho, The double shuffle relations for p-adic multiple zeta values, Contemp. Math., **416**, Amer. Math. Soc., Providence, RI, 2006.

[C1] K.T.Chen, Reduced bar constructions on de Rham complexes, Algebra, topology, and category theory (a collection of papers in honor of Samuel Eilenberg), pp. 19–32. Academic Press, New York, 1976.

[C2] K.T.Chen, Reduced bar constructions on de Rham complexes, Algebra, topology, and category theory (a collection of papers in honor of Samuel Eilenberg), pp. 19–32. Academic Press, New York, 1976.

[C3] K.T.Chen, Iterated path integrals, Bull. Amer. Math. Soc. **83** (1977), no.5 831–879.

[D] V.G.Drinfel'd, On quasitriangular quasi-Hopf algebras and on a group that is closely connected with Gal($\overline{\mathbf{Q}}/\mathbf{Q}$), Algebra i Analiz 2 (1990), no. 4, 149–181; translation in Leningrad Math. J. **2** (1991), no. 4, 829–860.

[DT] P.Deligne and T.Terasoma, Harmonic shuffle relation for associators, preprint (2005), `http://www2.lifl.fr/mzv2005/DOC/Terasoma/lille_terasoma.pdf`.

[F] H.Furusho, Double shuffle relation for associators, preprint (2008) arXiv:math.AG/0808.0319v1.

[G] A.B.Goncharov, Multiple polylogarithms and mixed Tate motives, preprint (2001) arXiv:math.AG/010359.

[Ha] R.M.Hain, Iterated integrals and mixed Hodge structures on homotopy groups, Hodge theory (Sant Cugat, 1985), 75–83, Lecture Notes in Math., 1246, Springer, Berlin, 1987.

[Ho] M.E.Hoffman, The algebra of multiple harmonic series, J. Algebra 194 (1997), no. 2, 477–495.

[I] Y.Ihara, Automorphisms of pure sphere braid groups and Galois representations, The Grothendieck Festschrift, Vol. II, 353–373, Progr. Math., 87, Birkhäuser Boston, Boston, MA, 1990.

[IKZ] K.Ihara, M.Kaneko and D.Zagier, Derivation and double shuffle relations for multiple zeta values, Compos. Math. 142 (2006), no. 2, 307–338.

[K] C. Kassel, Quantum Groups, Graduate Texts in Mathematics, 155, Springer-Velag, (1995).

[LM] T.Q.T.Le and J.Murakami, Kontsevich's integral for the Homfly polynomial and relations between values of multiple zeta functions, Topology Appl. 62 (1995), no. 2, 193–206.

[OU1] S.Oi and K.Ueno, Connection problem of the formal KZ equation of two variables and the five term relation of dilogarithm, in preparation.

[OU2] S.Oi and K.Ueno, Formal KZ equation on $\mathcal{M}_{0,n}$, in preparation.

[OkU] J.Okuda and K.Ueno, The Sum Formula for Multiple Zeta Values, Zeta Functions, Topology and Quantum Physics, Developments in Mathematics 14, ed. by T. Aoki et al., Springer-Verlag (2005), 145-170.

[OS] P.Orlik and L.Solomon, Combinatorics and topology of complements of hyperplanes, Invent. Math. 56 (1980), no. 2, 167–189.

[OT] P.Orlik and H.Terao, Arrangements of Hyperplanes, A Series of Comprehensive Studies in Mathematics 300, Springer-Verlag, Berlin, 1992.

[R] C.Reutenauer, Free Lie Algebras, Oxford Science Publications,1993.

[W] Z.Wojtkowiak, Monodromy of iterated integrals and non-abelian unipotent periods, Geometric Galois actions, 2, 219–289, London Math. Soc. Lecture Note Ser., 243, Cambridge Univ. Press, Cambridge, 1997.